# S6 ERRORS PART ONE 2020

Whenever an estimate of a value is made, an error may be made. The measure of the difference between the exact value and the approximated value constitutes the error. So an error can either be positive or negative.

**Definition**

$$Error = Exact\,value - Approximated\,value$$

Suppose that the number $X$ is estimated to $x$ with an error $\Delta x$, then

$$Error = \Delta x = X - x \Rightarrow X = x + \Delta x.$$

Note: In mathematics, errors are always made due to **rounding off** or when a finite value is truncated.

Example

*Round off the following to three significant figures.*

*(i) 0.48246 (ii) 0.048258 (iii) 0.0100985(iv) 89.7846 (v) 9.9999 (vi) 48564*

Answers

(i) 0.482 (ii) 0.0483 (iii) 0.0101 (iv) 89.8 (v) 10.0 (vi) 48600

Note : You should note that in each of the previous examples, there is a difference between the actual value and the estimated value.

Truncation – This refers to termination of a given value or observation at a certain stage.

Example

*Truncate the following to three significant figures.*

*(a) 0.48346 (b) 0.0049468 (c) 4.28946 (d) 9.8989 (e) 0.900946 (f) 848456*

*(g) 1999857*

Answers

(a ) 0.483 (b) 0.00494 (c) 4.28 (d) 9.89 (e) 0.900 (f) 848000 (g) 1990000

Note that in part (f) and (g), the zeros stand for place values of thousands.

**Exercise**

1. Round off the following to three significant figures.

   (a) 0.4949 (b) 8.99425 (c) 0.04899 (d) 0.001058 (e) 10.9090 (f) 845.48 (g) 9.9909 (h) 14.987

2. Truncate the following to two significant figures.

   (a) 0.004949 (b) 9.09425 (c) 7.77899 (d) 40984 (e) 122222 (f) 845.48 (g) 9.9909 (h) 989884

TWRMS USED IN ERRORS

(1) $Error = Exact\ value - Approximated\ value$

(2) Absolute error $= |Error|$ i.e. take only the positive value / size of the error.

(3) Relative error = the positive value of the ratio of the error to the exact value i.e.

$$Relative\ error = \left|\frac{Error}{Exact\ value}\right|$$

(4) Percentage error = $Relative\ error \times 100 = \left|\frac{Error}{Exact\ value}\right| \times 100$ .

**Note**: *It should generally be noted that when a number (value) is rounded off to n decimal places, the possible error made is given by* $\pm 0.5 \times 10^{-n}$.

For example, for the respective numbers below, the possible errors are given as;

(i) A = 0.4 , $e_A = \Delta A = \pm 0.05$

(ii) B = 0.48 , $e_B = \pm 0.005$

(iii) C = 9 , $e_C = \pm 0.5$

(iv) D = - 4.845 , $e_D = \pm 0.0005$

From the above observations, we can deduce the maximum and minimum values of each observation. That is to say,

Maximum value = Value + Error

Minimum value = Value – Error

**Maximum and Minimum values of expressions**

Suppose that the numbers A and B have been estimated with errors $\Delta A$ and $\Delta B$ respectively, then for;

(a) Sum/ Addition

$$(A+B)_{\max} = A_{\max} + B_{\max} = (A+\Delta A)+(B+\Delta B)$$

$$(A+B)_{\min} = A_{\min} + B_{\min} = (A-\Delta A)+(B-\Delta B)$$

(b) Difference / Subtraction

$(A-B)_{\max} = A_{\max} - B_{\min} = (A+\Delta A)-(B-\Delta B)$ i.e subtract off the minimum of the second term.

$(A-B)_{\min} = A_{\min} - B_{\max} = (A-\Delta A)-(B+\Delta B)$ i.e subtract off the maximum of the second term.

(c) Product

$$(AB)_{\max} = A_{\max} \times B_{\max} = (A+\Delta A)\times(B+\Delta B)$$

$$(AB)_{\min} = A_{\min} \times B_{\min} = (A-\Delta A)\times(B-\Delta B)$$

(d) Quotient/ Division

$\left(\frac{A}{B}\right)_{\max} = \frac{A_{\max}}{B_{\min}} = \frac{A+\Delta A}{B-\Delta B}$ i.e divide by smallest value of B.

$\left(\frac{A}{B}\right)_{\min} = \frac{A_{\min}}{B_{\max}} = \frac{A-\Delta A}{B+\Delta B}$ i.e divide by biggest vale of B.

NOTE: We can use the maximum and minimum values of an expression to obtain the maximum error that can be made in an expression. That is to say

$$Error = \frac{Max.value - Min.value}{2}$$

Example 1

*The numbers* $P = 4.8$ , $Q = 5.25$ *and* $R = 13$ *are rounded off to the given number of decimal places. Find the error made in each of the following expressions.*

| | | | | | |
|---|---|---|---|---|---|
| (a) | $P + Q$ | (b) | $Q - P$ | (c) | $PQ$ |
| (d) | $\dfrac{P}{Q}$ | (e) | $\dfrac{PQ}{R}$ | (f) | $\dfrac{Q - P}{R}$ |
| (g) | $\dfrac{P}{R - Q}$ | (h) | $P - \dfrac{R}{Q}$ | (i) | $\dfrac{R}{P} - \dfrac{P}{Q}$ |

Solution

$P = 4.8$ , $\Delta P = \pm\, 0.05$ ; $Q = 5.25$ , $\Delta Q = \pm 0.005$ ; $R = 13$ , $\Delta R = \pm 0.5$

(a) $P + Q$

$$(P + Q)_{\max} = P_{\max} + Q_{\max} = 4.85 + 5.255 = 10.105$$

$$(P + Q)_{\min} = P_{\min} + Q_{\min} = 4.75 + 5.245 = 9.995$$

$$Error = \frac{10.105 - 9.995}{2} = 0.055$$

(b) $Q - P$

$$(Q - P)_{\max} = Q_{\max} - P_{\min} = 5.255 - 4.75 = 0.505$$

$$(Q - P)_{\min} = Q_{\min} - P_{\max} = 5.245 - 4.85 = 0.395$$

$$Error = \frac{0.505 - 0.395}{2} = 0.055$$

(c) $PQ$

$$(PQ)_{\max} = P_{\max} \times Q_{\max} = 4.85 \times 5.255 = 25.4668$$

$$(PQ)_{\min} = P_{\min} \times Q_{\min} = 4.75 \times 5.245 = 24.9138$$

$$Error = \frac{25.4668 - 24.9138}{2} = 0.2765$$

(d) $\frac{P}{Q}$

$$\left(\frac{P}{Q}\right)_{\max} = \frac{P_{\max}}{Q_{\min}} = \frac{4.85}{5.245} = 0.9247$$

$$\left(\frac{P}{Q}\right)_{\min} = \frac{P_{\min}}{Q_{\max}} = \frac{4.75}{5.255} = 0.9039$$

$$Error = \frac{0.9247 - 0.9039}{2} = 0.0104$$

(e) $\frac{PQ}{R}$

$$\left(\frac{PQ}{R}\right)_{\max} = \frac{P_{\max} \times Q_{\max}}{R_{\min}} = \frac{4.85 \times 5.255}{12.5} = 2.0383$$

$$\left(\frac{PQ}{R}\right)_{\min} = \frac{P_{\min} \times Q_{\min}}{R_{\max}} = \frac{4.75 \times 5.245}{13.5} = 1.8455$$

$$Error = \frac{2.0383 - 1.8455}{2} = 0.0964$$

(f) $\frac{Q - P}{R}$

$$\left(\frac{Q-P}{R}\right)_{\max} = \frac{(Q-P)_{\max}}{R_{\min}} = \frac{5.255 - 4.75}{12.5} = 0.0404$$

$$\left(\frac{Q-P}{R}\right)_{\min} = \frac{(Q-P)_{\min}}{R_{\max}} = \frac{5.245 - 4.85}{13.5} = 0.0293$$

$$Error = \frac{0.0404 - 0.0293}{2} = 0.00555$$

(g) $\frac{P}{R - Q}$

$$\left(\frac{P}{R-Q}\right)_{\max} = \frac{P_{\max}}{(R-Q)_{\min}} = \frac{4.85}{12.5-5.255} = 0.6694$$

$$\left(\frac{P}{R-Q}\right)_{\min} = \frac{P_{\min}}{(R-Q)_{\max}} = \frac{4.75}{13.5-5.245} = 0.5754$$

$$Error = \frac{0.6694 - 0.5754}{2} = 0.0470$$

(h) $P - \frac{R}{Q}$

$$\left(P - \frac{R}{Q}\right)_{\max} = P_{\max} - \frac{R_{\min}}{Q_{\max}} = 4.85 - \frac{12.5}{5.255} = 2.4713$$

$$\left(P - \frac{R}{Q}\right)_{\min} = P_{\min} - \frac{R_{\max}}{Q_{\min}} = 4.75 - \frac{13.5}{5.245} = 2.1761$$

$$Error = \frac{2.4713 - 2.1761}{2} = 0.1476$$

(i) $\frac{R}{P} - \frac{P}{Q}$

$$\left(\frac{R}{P} - \frac{P}{Q}\right)_{\max} = \left(\frac{R}{P}\right)_{\max} - \left(\frac{P}{Q}\right)_{\min} = \frac{13.5}{4.75} - \frac{4.75}{5.255} = 1.9382$$

$$\left(\frac{R}{P} - \frac{P}{Q}\right)_{\min} = \left(\frac{R}{P}\right)_{\min} - \left(\frac{P}{Q}\right)_{\max} = \frac{12.5}{4.85} - \frac{4.85}{5.245} = 1.6526$$

$$Error = \frac{1.9382 - 1.6526}{2} = 0.1428$$

Example 2

*The numbers a = 4.5 and b = 1.24 are estimated with relative errors of 0.01 and 0.05 respectively. Find the error made in estimating* $\frac{a}{(a-b)^2}$.

Solution

In this case we use the definition of relative error to obtain the error made in each of the terms.

$$\left|\frac{\Delta a}{a}\right| = 0.01 \Rightarrow |\Delta a| = 0.01 \times 4.5 = 0.045$$

$$\left|\frac{\Delta b}{b}\right| = 0.05 \Rightarrow |\Delta b| = 0.05 \times 1.24 = 0.062$$

Now, $a_{\max} = 4.545$ ; $a_{\min} = 4.455$ ; $b_{\max} = 1.302$ ; $b_{\min} = 1.178$

$$\left(\frac{a}{(a-b)^2}\right)_{\max} = \frac{a_{\max}}{(a-b)^2_{\min}} = \frac{4.545}{(4.455 - 1.302)^2} = 0.4572$$

$$\left(\frac{a}{(a-b)^2}\right)_{\min} = \frac{a_{\min}}{(a-b)^2_{\max}} = \frac{4.455}{(4.545 - 1.1.178)^2} = 0.3930$$

$$Error = \frac{0.4572 - 0.3930}{2} = 0.0321$$

Example 3

*The numbers m = 4 and n = 5.8 are estimated with percentage errors of 0.1% and 0.05% respectively. Calculate the percentage error made in the expression* $\frac{m+n}{n-m}$.

Solution

In this question, use the definition for percentage error to obtain the error made in each of *m* and *n*.

$$\left|\frac{\Delta m}{m}\right| \times 100 = 0.1 \Rightarrow |\Delta m| = \frac{0.1}{100} \times 4 = 0.004$$

$$\left|\frac{\Delta n}{n}\right| \times 100 = 0.05 \Rightarrow |\Delta n| = \frac{0.05}{100} \times 5.8 = 0.0029$$

Now, $m_{\max} = 4.004$ ; $m_{\min} = 3.996$ ; $n_{\max} = 5.8029$ ; $n_{\min} = 5.7971$

$$\left(\frac{m+n}{n-m}\right)_{\max} = \frac{(m+n)_{\max}}{(n-m)_{\min}} = \frac{4.004 + 5.8029}{5.7971 - 4.004} = 5.4692$$

$$\left(\frac{m+n}{n-m}\right)_{\min} = \frac{(m+n)_{\min}}{(n-m)_{\max}} = \frac{3.996+5.7971}{5.8029-3.996} = 5.4198$$

$$Error = \frac{5.4692-5.4198}{2} = 0.0247$$

Working value = $\frac{m+n}{n-m} = \frac{4+5.8}{5.8-4} = 5.4444$ (obtained by substituting the values of $m$ and $n$ given in the question)

$$\%ge\ error = \left|\frac{error}{working\ value}\right| \times 100 = \frac{0.0247}{5.4444}\times 100 = 0.4537\%$$

Note:

(i) The range of values of an expression can be stated as $[Min.\ value\ ,\ Max.value]$ or $Min.\ value \leq True\ value \leq Max.\ value$.

(ii) If required to state the limits within which the exact value of an expression lies, then list them individually, i.e.
Lower Limit = Minimum value = Least value
Upper Limit = Maximum value = Greatest value

Example 4

*The length, breadth and height of a metallic tank were measured as 4.2 m, 3 m and 4.25 m respectively, to the given number of decimal places.*

*(a) State the possible error made in each of the length, breadth and height.*
*(b) Calculate the range within which the volume of the tank lies.*

Solution

(a) Length, $l$ = 4.2; $\Delta l = \pm\ 0.05$, Breadth, $b$ = 3; $\Delta b = \pm 0.5$, height, $h$ = 4.25; $\Delta h = \pm 0.005$

(b) Volume, $V = l \times b \times h$

$$V_{\max} = l_{\max} \times b_{\max} \times h_{\max} = 4.25 \times 3.5 \times 4.255 = 63.2931\ m^3$$

$$V_{\min} = l_{\min} \times b_{\min} \times h_{\min} = 4.15 \times 2.5 \times 4.245 = 44.0439\ m^3$$

Range = $[44.0439\ ,\ 63.2931]$

**Maximum and Minimum values of trigonometric functions**

Suppose that $\Delta\theta$ is the error made in measuring the angle, $\theta$, then;

(a) $(\sin\theta)_{\max} = \sin(\theta+\Delta\theta)$ and $(\sin\theta)_{\min} = \sin(\theta-\Delta\theta)$

(b) $(\cos\theta)_{\max} = \cos(\theta-\Delta\theta)$ and $(\cos\theta)_{\min} = \cos(\theta+\Delta\theta)$

(c) $(\sec\theta)_{\max} = \dfrac{1}{(\cos\theta)_{\min}} = \dfrac{1}{\cos(\theta+\Delta\theta)}$ and $(\sec\theta)_{\min} = \dfrac{1}{(\cos\theta)_{\max}} = \dfrac{1}{\cos(\theta-\Delta\theta)}$

(d) $(\cos ec\,\theta)_{\max} = \dfrac{1}{(\sin\theta)_{\min}} = \dfrac{1}{\sin(\theta-\Delta\theta)}$ and $(\cos ec\,\theta)_{\min} = \dfrac{1}{(\sin\theta)_{\max}} = \dfrac{1}{\sin(\theta+\Delta\theta)}$.

Example 5

*Given that* $y = \sec\theta$ *and* $\theta = 15 \pm 0.1^0$*, find the limits within which the exact value of y lies.*

Solution

$\theta = 15^0$ ; $\Delta\theta = \pm\, 0.1^0$

Lower limit = $(\sec\theta)_{\min} = \dfrac{1}{\cos(15-0.1)} = 1.0348$

Upper Limit = $(\sec\theta)_{\max} = \dfrac{1}{\cos(15+0.1)} = 1.0358$

Example 6

*The area of a triangle with adjacent sides a and b and included angle* $\theta$*, is calculated using the formula* $A = \dfrac{1}{2}ab\sin\theta$*. Given that* $a = 6.2 \pm 0.25\ cm$, $b = 4.4 \pm 0.05\ cm$ *and* $\theta = 25 \pm 0.5^0$*, find the limits within which the true value of the area lies.*

Solution

$a = 6.2$ , $\Delta a = \pm\, 0.25$ ; $b = 4.4$ , $\Delta b = \pm\, 0.05$ ; $\theta = 25^0$ , $\Delta\theta = \pm\, 0.5^0$

Lower Limit = $A_{\min} = \dfrac{1}{2}a_{\min} \times b_{\min} \times (\sin\theta)_{\min}$

$= \dfrac{1}{2} \times 5.95 \times 4.35 \times \sin 24.5^0 = 5.3666\ cm^2$

Upper Limit $= \quad A_{\max} = \frac{1}{2} a_{\max} \times b_{\max} \times (\sin\theta)_{\max}$

$$= \frac{1}{2} \times 6.45 \times 4.45 \times \sin 25.5^0 = 6.1784\ cm^2$$

Example 7

*Given that the numbers* $a = -4.8$ , $b = 2.54$ *and* $c = 9$ *are rounded off to the given number of decimal places, find the limit within which the exact values of the following expressions lie.*

*(a)* $a(b+c)$ *(b)* $\frac{a}{b+c}$ *(c)* $\frac{c}{a-b}$

Solution

$a = -4.8$ , $\Delta a = \pm 0.05$ ; $b = 2.54$ , $\Delta b = \pm\ 0.005$ ; $c = 9$ , $\Delta c = \pm\ 0.5$

In each of the following parts, first obtain the working value. If the working value is negative, then there is need for critical thinking, i.e. a bigger negative will be the minimum value whereas the smaller negative will be the maximum value.

(a) $a(b+c)$

Working value = $a(b+c) = -4.8\ (2.54 + 9) = -55.392$

Lower Limit = $-[4.8(2.54 + 9)]_{\max} = -4.85(2.545 + 9.5) = -58.41825$

Upper Limit = $-[4.8(2.54 + 9)]_{\min} = -4.75(2.535 + 8.5) = -\ 52.41625$

(b) $\frac{a}{b+c}$

$\frac{a}{b+c} = \frac{-4.8}{2.54+9} = -\left(\frac{4.8}{2.54+9}\right)$ i.e. factorise a negative sign and then analyse.

Lower Limit = $-\left(\frac{4.8}{2.54+9}\right)_{\max} = -\left(\frac{4.85}{2.535+8.5}\right) = -0.4395$

Upper Limit = $-\left(\frac{4.8}{2.54+9}\right)_{\min} = -\left(\frac{4.75}{2.545+9.5}\right) = -0.3944$

(c) $\frac{c}{a-b}$

$$\frac{c}{a-b} = \frac{9}{-4.8-2.54} = -\left(\frac{9}{4.8+2.54}\right)$$ i.e. factorise a negative from the denominator.

Lower Limit = $-\left(\frac{9}{4.8+2.54}\right)_{max} = -\frac{9.5}{4.75+2.535} = -1.3040$

Upper Limit = $-\left(\frac{9}{4.8+2.54}\right)_{min} = -\frac{8.5}{4.85+2.545} = -1.1494$

**EXERCISE**

1. The numbers $A$ = 12.31 and $B$ = 6.241 are rounded off to the given number of decimal places. Find the maximum error made in the following expressions (a) $A + B$ (b) $A - B$ (c) $AB$ (d) $\frac{A}{B}$. Ans: 0.0055, 0.0055 , 0.0374 , 0.00096
2. The numbers $P$ = 2.41, $q$ = 1.23 and $r$ = 2.0 have been rounded off to the given number of decimal places. Calculate to four significant figures the limits within which the exact value of $P - \frac{q}{r}$ lies. Ans: 1.772, 1.817
3. Given that $x$ = 5.73, $y$ = -2.496 and $z$ = 5.9765 are rounded off to the given number of decimal places, find correct to three significant figures, the maximum and minimum values of $\frac{x}{y-z}$.     Ans: - 0.677, - 0.676
4. The radius and height of a cone are measured as 4.7 cm and 12.65 cm with errors of 0.3 and 0.45 cm respectively. Calculate the interval within which the volume of the cone lies. Ans: 247.3397, 342.9572
5. Given that the numbers w = 28.114, x = 7.136 , y 41.84464 and z = 3.6827 are estimated to the given number of decimal places, find the percentage of the error made in $\frac{w}{x} - \frac{y}{z}$. Ans: 0.0074
6. The numbers A = 12.4, B = 29.45 and C = 4.25 are rounded off with percentage errors of 2%, 0.2% and 1%, find the limits within which the exact value of $\frac{A}{(B-C)^2}$ lies.
7. Calculate the range within which the exact value of $5.2\sqrt{2.85}$ lies.
8. The numbers $x = 4.2$, $y = 16.02$, $and\ z = 25$ are rounded off with corresponding percentage errors of 0.5, 0.45 and 0.02, calculate the absolute relative error made in $\frac{x+y}{z}$.
9. Given that y = sinθ and θ is measured with maximum possible error of 2%. If $\theta = 30^o$, determine the: (i) absolute error in y (ii) interval within which the value of y lies.
10. Given that $P$ = 4.8 and $Q$ = 21.32 are rounded off with percentage errors 0.2 and 0.06 respectively. Find the percentage error in the numbers $\frac{P}{Q}$.