MATHEMATICS
Encyclopaedia

From basics (age 11) up to GCE
Advanced Subsidiary Level (Age 17)

By R.M. O'Toole
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Encyclopaedia of Mathematics

From basics (age 11) to GCE Advanced Subsidiary – most of material extracted from our books – some new.

Mathematics Encyclopaedia

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First Published in 1996 (Revised in 2006) by:

Mathematics Publishing Company,
'The Cottage'
45, Blackstaff Road,
Clough,
Downpatrick,
Co. Down,
BT30 8SR.

Tel. 028 44 851818 Fax. 028 44 851818

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ISBN No. 1900043 70X

Testimonials

Below is a selection of written comments received from people who have used our material.

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Extracts from a head teacher's letter:

'*… very well received by parents, teachers and pupils …*'
'*… self contained…*'
'*… highly structured …*'
'*… all children including the less well able are helped …*'
'*…to develop concepts through a series of clearly defined steps …*'
'*… increased confidence for pupils …*'
'*… parents find user friendly as worked examples are given …*'
'*… language and notation are simple and clearly defined …*'

From a 10 year old pupil (boy):

'*… the material describes the working out in a way that is easy. The worked examples are laid out very clearly …*'

GCSE (15-16 year olds)

From a 15 year old pupil (boy):

'*… simple and easy way to learn maths …*'
'*… careful explanations of each topic…*'
'*… also questions to make sure you know and understand what you have learned, and each question has a worked answer to check everything you have covered …*'
'*… so you are never left without any help …*'

From a 15 year old pupil (girl):

'*…easy to understand …*'
'*…clear and concise …*'
'*…thoroughly recommended …*'

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From a 16 year old pupil (girl):

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‘… *clearly explained and easy to understand* …’
‘… *well laid out* …’
‘… *well structured* …’
‘… *I would not hesitate to use these again* …’

From a 16 year old pupil (boy):

‘… *self-explanatory and easy* …’
‘… *laid down basis of skill required* …’
‘… *helped me consolidate* …’
‘… *succinct and effective* …’
‘… *boosted my confidence* …’
‘… *contributed significantly towards helping me to prepare for exams* …’

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From a 19 year old university student (man):

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‘… *your material… gave me help and reinforcement* …’
‘… *increasing my confidence to pursue my maths* …’
‘… *I am now enjoying life at university* …’

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Acknowledgements

Special thanks to Cormac O'Toole and Denise Tumelty who advised on the computer technical end.

Finally, I would like to thank all our customers for buying our books and for their kind letters of appreciation.

G.B. O'Toole, B.A. (Hons.), CertPFS

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Dedication

I dedicate this work to my grandchildren, David, Eoin, Aidan, Rory, Adam and Lucy.

Ros.

FOREWORD

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The entries in the Encyclopaedia are either from our books on the website or were written specifically for the Encyclopaedia by our author.

If you would like more detailed information on a topic over and above that which is contained in the Encyclopaedia – then you should go to the related material on www.mathslearning.com, covering ages from **eleven** up to **eighteen** - Key Stage 2, Key Stage 3, GCSE (Ordinary), GCSE (Additional), AS and A Level – also 'Algebra – the way to do it', a special book that will teach you algebra – all available from www.mathslearning.com for instant download to your computer.

Of necessity, each entry in the Encyclopaedia is a condensed version of the material contained on www.mathslearning.com. The material in the Encyclopaedia covers age groups from **eleven** (Key Stage 2) to around **seventeen** years (Advanced Subsidiary) and should prove extremely useful for exam revision.

Mathematics is now a requirement for most careers, or, as Roger Bacon put it in 1267 : 'mathematics is the gate and key of the sciences'.

Algebra – there are some algebraic entries in the Encyclopaedia. Since algebra poses a particular problem, our author has written a separate publication called "Algebra – the way to do it" **-** this is also available on the website. Our algebra publication will take you step-by-step from the beginning right up through to Advanced Subsidiary level. It does not matter if you are in the U.K., the U.S.A., Ireland or Iceland, Australia or Africa, algebra is the same the world over. Algebra is an important part of mathematics. When you master it, you will have a real sense of achievement; resolve to do that.

Practice Papers **-** the Sections on www.mathslearning.com contain teaching text, worked examples and exercises with full, worked answers.
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Good luck in your studies.

The Publisher.

Acceleration (Mechanics – GCSE Additional and Advanced Subsidiary)

Acceleration is the rate of change of **velocity** with **time**, generally in metres per second per second, $\mathbf{m/s}^2$ or $\mathbf{ms}^{-2}$.

A force of **F** newtons acting on a body of mass **m** kg produces an **acceleration** of **a** m/s^2, giving the **equation** of **motion**:

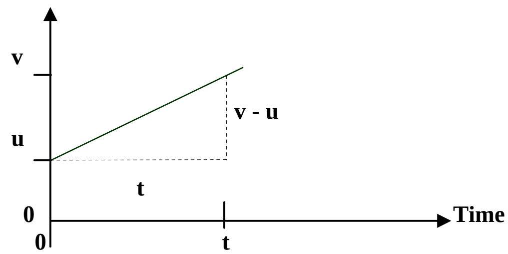
$$\mathbf{F} = \mathbf{ma}.$$

E.g. A **force** of **5N** acting on a body of **mass 10kg** produces an **acceleration** of $\mathbf{0.5m/s^2}$:

$$\mathbf{5} = \mathbf{10(0.5)}$$

- The **gradient** of a **velocity/time graph** gives **acceleration**:

E.g. **Velocity** (**Constant Acceleration**)



Let **u** be the **initial velocity** and **v** be the **velocity at time t**.

$$\mathbf{a} = \frac{\mathbf{v - u}}{\mathbf{t}}$$

$\Rightarrow$ $\mathbf{at} = \mathbf{v - u}$ **(By cross-multiplication)**

or $\mathbf{v} = \mathbf{u + at}$.

When the acceleration is **variable**, the **velocity**/**time** graph is **curved**.
In this case, the gradient may be **approximated** by drawing a **tangent** to the curve at the required **time point** and calculating its **gradient**.

- The accurate way to find the **acceleration** is to **differentiate** **velocity** with respect to **time** i.e. **a** (acceleration) $= \dfrac{\mathbf{dv}}{\mathbf{dt}}$;

- not always an option as it can be used only when an equation of **velocity** in terms of **time** is known.

E.g. The velocity, **v**, of a moving body is $(\mathbf{2t^2 + 1})$m/s, after a time, **t** seconds. Find the **acceleration** after **1** second.

$$\mathbf{a} = \frac{\mathbf{dv}}{\mathbf{dt}} = \mathbf{4t}$$

$\mathbf{t = 1}$ $\Rightarrow$ $\mathbf{a} = \mathbf{4m/s^2}$ after **1** second.

- The **area** under an **acceleration**/**time** graph gives the **velocity**. This area may be found by using one of the **area-approximating rules** (Mid-ordinate, Trapezium or Simpson's).

- The accurate way to find the **velocity** is to **integrate acceleration** with respect to **time** i.e. **v** (velocity) **=** $\int_0 a\,dt + c$ **-** not always an option as it can be used only when an equation of **acceleration** in terms of **time** is known.

  E.g. The **acceleration, a**, of a moving body at the end of **t** seconds from the start of motion is **(7 – t)m/s** ^{2}. Find the **velocity** at the end of **3 seconds** if the **initial velocity** is **5m/s**.

$$v = \int_0 a\,dt + c$$

$$\Rightarrow \quad v = \int_0 (7 - t)\,dt + c$$

$$\Rightarrow \quad v = 7t - \frac{t^2}{2} + c.$$

$$t = 0,\ v = 5 \Rightarrow 5 = 7(0) - \left(\frac{0^2}{2}\right) + c$$

$$\Rightarrow \quad c = 5$$

$$\therefore \quad v = 7t - \frac{t^2}{2} + 5.$$

$$t = 3 \quad \Rightarrow \quad v = 7(3) - \left(\frac{3^2}{2}\right) + 5$$

$$\therefore \quad v = 21\frac{1}{2}\ \text{m/s} \text{ after } 3 \text{ seconds.}$$

Algebra

The word **'algebra'** comes from the Arabic word **'algorithm'**, meaning **'a step-by-step process for performing calculations'**.
Algebra is a **special kind of arithmetic** that uses **letters (or symbols)** instead of numbers to represent quantities.

The only difference is that **x**, for example, can stand for **any quantity**, whereas a **number** like **3**, for example, **stands only** for a **set of three things**.
In calculations, **x** is used in exactly the same way as **3**, or **any** other number.

The **four basic rules**, namely:
addition, subtraction, multiplication and **division**,
are applied in **algebra** in the **same way** as they are in **arithmetic**.

<u>Addition</u>

x + 3 means **3** is **added to x**.
x + 3x means **1 of x** is added to **3 of x**, giving **4 of x** altogether.
We call this **4x**.

Note the difference between **x + 3** and **x + 3x**.
x + y means a quantity ***x*** is added to a quantity ***y***.
Note that **x + y** is the **same** as **y + x**.
Since the **order is not important** we say that quantities are **commutative under addition**.

E.g.(i) Add 2x – 3 and x + 5.
2x – 3 + x + 5 = 3x +2.

E.g. (ii) Add ¾ x^2 and ½ x^3.

$$\frac{3x^2}{4}+\frac{x^3}{2}=\frac{3x^2}{4}+\frac{2x^3}{4}=\frac{3x^2+2x^3}{4}=\frac{x^2(3+2x)}{4}.$$

Subtraction

3 - x means **x is subtracted from 3**.
3x - x means **1 of x** is subtracted from **3 of x**, leaving **2 of x**.
We call this **2x**.
Note the **difference** between **3 - x** and **3x - x**.
Also: **x - y** is **not the same** as **y – x**.
Since the **order is important**, we say that quantities are **not commutative under substraction**.

E.g.(i) Subtract 2x – 3 from x + 5.
x + 5 – (2x – 3) = x + 5 – 2x + 3.

E.g. (ii) Subtract ½ x^3 from ¾ x^2.

$$\frac{3x^2}{4}-\frac{x^3}{2}=\frac{3x^2}{4}-\frac{2x^3}{4}=\frac{3x^2-2x^3}{4}=\frac{x^2(3-2x)}{4}.$$

Multiplication

There is **no need to use a multiplication sign (×)** in algebra:
2x means **2 × x** or **x + x** and **5x** means **5 × x** or **x + x + x + x + x**.
As in arithmetic, **multiplication is a short method of addition**, where we have **2 ´ 9** short for **9 + 9** and **5 ´ 9** short for **9 + 9 + 9 + 9 + 9**.
xy means a quantity **x** is multiplied by another quantity **y**.
Note that **xy** is the **same** as **yx**.
Since the **order is not important**, we say that quantities are **commutative under multiplication**.

E.g.(i) Multiply (2x – 3) by (x + 5).
(x + 5)(2x – 3) = x(2x – 3) + 5(2x – 3)
= $2x^2$ - 3x + 10x – 15
= $2x^2$ + 7x – 15.

E.g. (ii) Multiply ¾ x^2 by ½ x^3.

$$\frac{3x^2}{4}\times\frac{x^3}{2}=\frac{3x^5}{8}.$$

Division

$\mathbf{\frac{x}{3}}$ means ***x*** is divided by **3**, giving $\mathbf{\frac{1}{3}}$ of **x**.

$\mathbf{\frac{x+1}{3}}$ means **1 is added to x** and this **result** is **divided by 3**.

$\frac{\mathbf{x}}{\mathbf{y}}$ means **x** is divided by **y** and $\frac{\mathbf{y}}{\mathbf{x}}$ means **y** is divided by **x**.

Since $\frac{\mathbf{x}}{\mathbf{y}}$ is **not the same** as $\frac{\mathbf{y}}{\mathbf{x}}$, as, for instance, **4** ¸ **2** is **not equal** to **2** ¸ **4**, we say that quantities are **not commutative under division**.

E.g. (i) Divide $2x^2 + 7x – 15$ by $(2x – 3)$.

$2x^2 + 7x – 15 = (x + 5)(2x – 3)$.

$\therefore (2x^2 + 7x – 15) \div (2x – 3) = (x + 5)$.

E.g. (ii) Divide ¾ x^2 by ½ x^3.

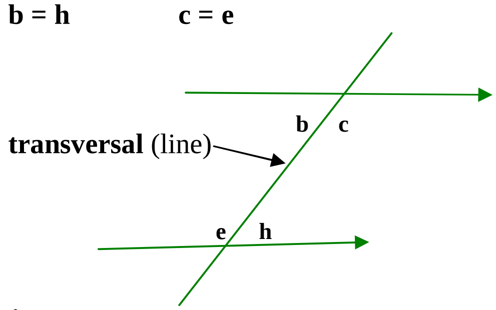
$$\frac{3x^2}{4} \div \frac{x^3}{2} = \frac{3x^2}{4} \times \frac{2}{x^3} = \frac{6x^2}{4x^3} = \frac{3}{2x}.$$

Alternate angle

Alternate angles are the angles on **either side** of a transversal drawn through a pair of **parallel** lines; these angles are **equal** to each other. (Latin: *alter* means *other*.)

Alternate angles in the diagram below:

b = h **c = e**



Approximation

It is useful to be able to **approximate** calculations.
Basic methods involve sensible guesswork and rounding off.
As a general rule, for every **numerator** that is made **larger** (or **smaller**), a **denominator** must be made **larger** (or **smaller**).
Decimal places, **significant figures**, and often, **standard form** are used in approximating.

Examples:

(i) $\frac{\mathbf{73 ´ 848}}{\mathbf{52 ´ 0.9}}$ » $\frac{\mathbf{70 ´ 850}}{\mathbf{50 ´ 1}}$ = $\frac{\mathbf{59500}}{\mathbf{50}}$ **= 1190 = 1000** to 1 sig. fig.

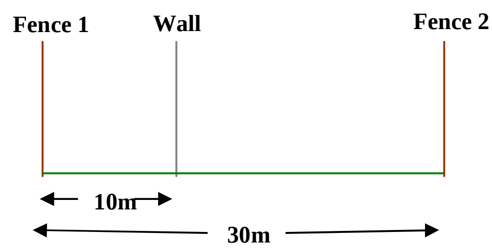
The exact answer is:
1322.735 which is **1000** to **1 sig**. **fig**.

(ii) $\frac{\mathbf{5.96 ´ (0.9)^2}}{\mathbf{3.15}}$ » $\frac{\mathbf{6 ´ 1^2}}{\mathbf{3}}$ = $\frac{\mathbf{6}}{\mathbf{3}}$ = **2** to 1 sig. fig.

The exact answer is:
1.533, giving **2** to **1 sig**. **fig**.

(iii) Two **parallel** fences on either side of a garden are **30m** apart, A light wall of negligible thickness is built in the garden, **10m** away from one of the fences.
(**Measurements** have been rounded off to the **nearest metre**.)

Find the **minimum** distance between the **wall** and the **second** fence.



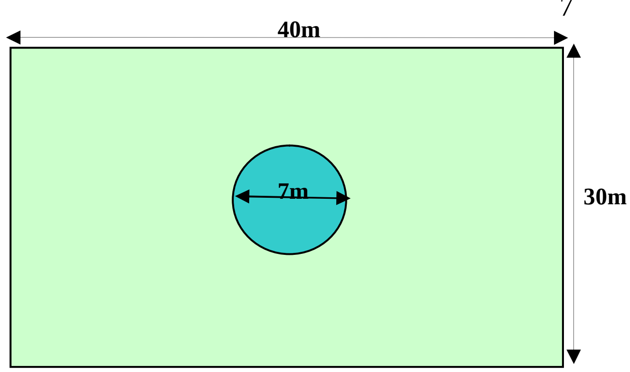
	Upper limit	**Lower limit**
30m	30.5m	29.5m
10m	10.5m	9.5m

Minimum distance between wall and fence 2:
29.5 – 10.5 = **19.5m**.

(iv) A rectangular plot of land of length **40m** and breadth **30m** contains a circular pond of diameter **7m** and the area remaining is planted with grass.
(Measurements have been rounded off to the nearest metre.)

Find (i) the **minimum** area of grass in the plot
and (ii) the **maximum** area of grass in the plot.
Round off your answers to three significant figures.

(Take π as $3\frac{1}{7}$).



	Upper limit	**Lower limit**
40m	40.5m	39.5m
30m	30.5m	29.5m
7m	7.5m	6.5m

Rectangle: **Minimum** area = 39.5 × 29.5 = 1165.25 m^2.
Maximum area = 40.5 × 30.5 = 1235.25 m^2.

Circle: **Minimum** area = $\pi \times (3\frac{1}{4})^2$ = 33.196428 m^2.

Maximum area = $\pi \times (3\frac{3}{4})^2$ = 44.196428 m^2.

Grass: **Minimum** area:
1165.25 – 44.196428 = 1121.0536 m^2.

Maximum area:
1235.25 – 33.196428 = 1202.0536 m^2.

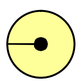
Answers to 3 significant figures:

Minimum area of grass: 1120 m^2.
Maximum area of grass: 1200 m^2.

Area

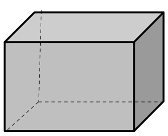
Area is **square measurement**, e.g. cm^2, m^2, etc.
It is found as a result of **multiplying 2 dimensions** together.

Area of this **rectangle**, **lb** = 2.5cm × 1cm = 2.5 cm^2.
(**l** = 2.5cm is length and **b** = 1cm is breadth).

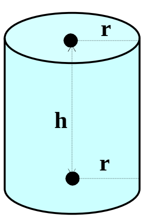


Area of this **circle**, $\pi r^2 = \pi \times 0.5^2 = 0.25\pi$ cm^2.

(**r** = 0.5cm is radius. **N.B.** π is **not** a dimension; $\pi \approx 3\frac{1}{7}$.)



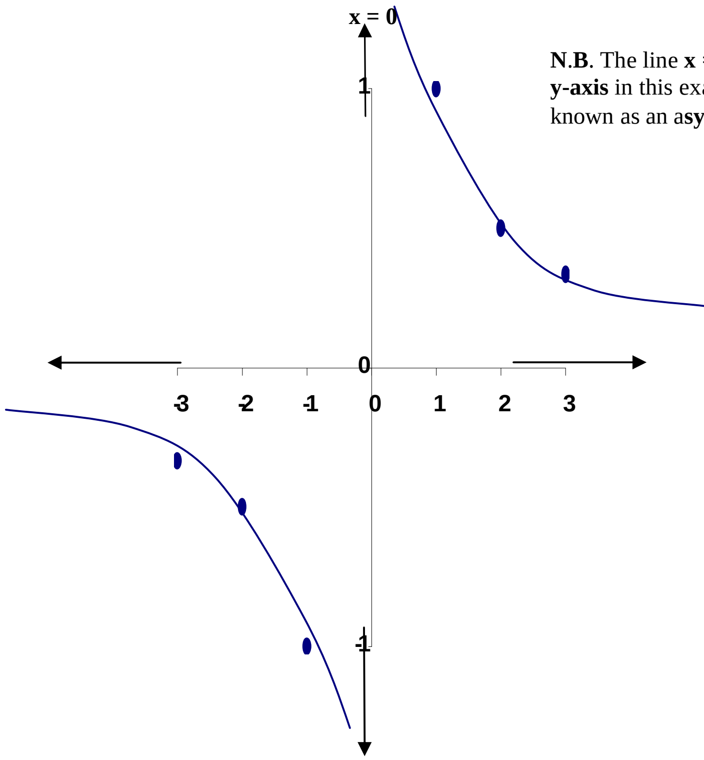
Surface Area of this **cuboid**, **2lb + 2lh + 2bh** = 23.5 cm^2.
(**l** = 2.5cm is length, **b** = 2cm is breadth and **h** = 1.5cm is height).



Surface Area of this **cylinder**, $2\pi r^2 + 2\pi rh$ = 22 cm^2.
(**r** = 1cm is radius, **h** = 2.5cm is height and $\pi = 3\frac{1}{7}$).

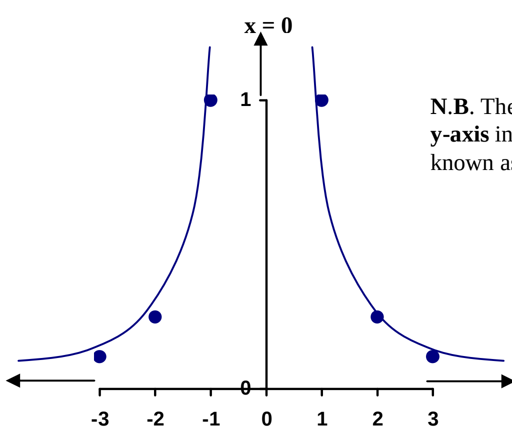
Asymptote The **lines** that the graphs *tend towards* but would only *actually touch* at infinity, namely **x = 0**, **x = 0** and **x = 1** in **Examples (i)**, **(ii)** and **(iii)** respectively are known as **asymptotes**.

(i) $y = \frac{1}{x}$, $(x \neq 0)$.



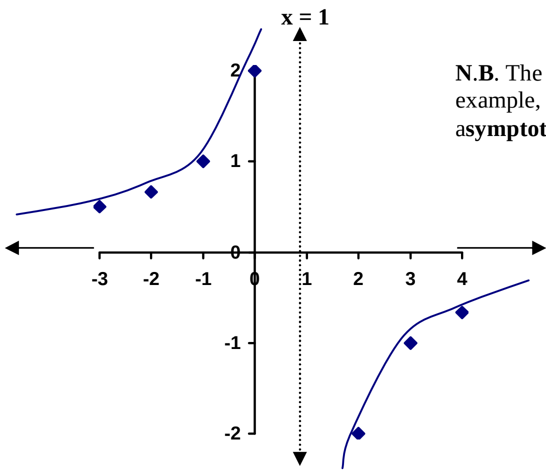
N.B. The line **x = o**, i.e. the **y-axis** in this example, is known as an a**symptote.**

(ii) $y = \frac{1}{x^2}$, $(x \neq 0)$.



N.B. The line **x = o**, i.e. the **y-axis** in this example, is known as an a**symptote.**

(iii) $y = \frac{2}{1 - x}$, $(x \neq 1)$.



N.B. The line **x = 1** in this example, is known as an a**symptote.**

Bar graph

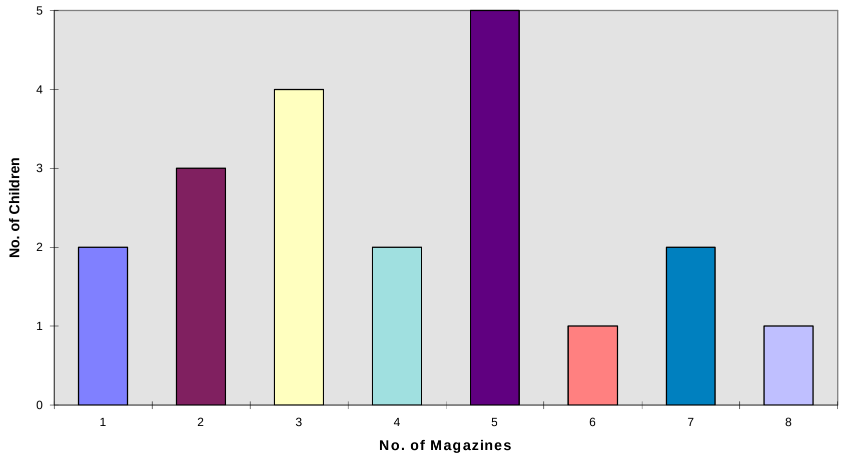
E.g. In a survey, **20** children were asked how many magazines they read in each week.
The results of the survey were as follows:

2	6	0	4	1
3	4	2	0	4
2	4	1	1	7
3	4	5	6	2

The results of this survey on magazines could be shown on a bar graph. In these graphs, the **data are represented by a series of bars, all of the same width**. The bars may be drawn **horizontally** or **vertically**. The **length** or **height** of **each bar** represents the **size of the figures**. The data that we need to portray are as follows:

	No. of Magazines	**No. of Children**
Bar 1	**0**	**2**
Bar 2	**1**	**3**
Bar 3	**2**	**4**
Bar 4	**3**	**2**
Bar 5	**4**	**5**
Bar 6	**5**	**1**
Bar 7	**6**	**2**
Bar 8	**7**	**1**

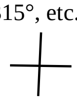
BAR GRAPH



Bearings

A **bearing** is the **direction**, measured in **degrees** from **North clockwise**, using **three figures**, to denote the position of something:

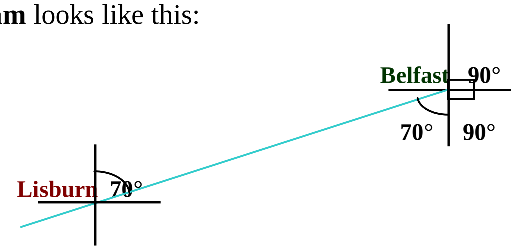
North is **000°**, **East** is **090°**, **South** is **180°**, **West** is **270°**, etc.
N.B. It follows that North–East is 045°, South–East is 135°, South–West is 135°, North–West is 315°, etc.

It is helpful to put a  around the **starting point** and around **each point** where there is a **change** of **bearing**.

It is very important to remember that **North** is **000°** and the **bearing** is measured from the **Northern Line** each time.

E.g. If Belfast is on a bearing of **070°** *from* Lisburn, calculate the **bearing** of Lisburn *from* Belfast.

The **diagram** looks like this:



The **bearing** of **Lisburn** from **Belfast** is, therefore:

90° + 90° + 70° = **250°**.

(N.B. Since **vertical** lines are **parallel** to each other, (as are **horizontal** lines), the **70°** angles are **alternate angles**.**)**

SAMPLE QUESTION – BEARINGS

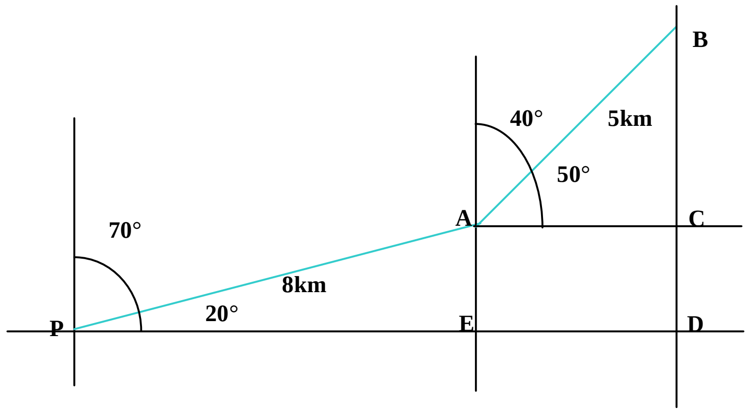
A boat sails **8km** from a port **P** on a bearing of **070°**.
It then sails **5km** on a bearing of **040°**.

Using **trigonometry**, find:

(i) the distance of the boat from P

(ii) the **bearing** of the boat from **P**.

The **diagram** looks like this:



Before we can use trigonometry, we must have **right-angled triangles**, so I have extended all the **vertical** and **horizontal** lines to achieve these. (**Remember that vertical and horizontal lines meet at right angles** .)

Since we need to find **PB**, we need to know **BD** and **PD**, and then we can apply **Pythagoras's Theorem**.

<u>Consider</u> <u>D</u> **<u>ABC</u>**<u>:</u>

$$\frac{\mathbf{Sin50°}}{\mathbf{1}} = \frac{\mathbf{BC}}{\mathbf{5}}$$

$\Rightarrow$ BC = 5 $\times$ Sin 50° (By cross-multiplication)

$\Rightarrow$ BC = 5 $\times$ 0.7660444

$\Rightarrow$ BC = 3.830 km.

$$\frac{\mathbf{Cos50°}}{\mathbf{1}} = \frac{\mathbf{AC}}{\mathbf{5}}$$

$\Rightarrow$ AC = 5 $\times$ Cos 50° (By cross-multiplication)

$\Rightarrow$ AC = 5 $\times$ 0.6427876

$\Rightarrow$ AC = 3.214 km.

<u>Consider</u> <u>D</u> **<u>APE</u>**:

$$\frac{\mathbf{Sin20°}}{\mathbf{1}} = \frac{\mathbf{AE}}{\mathbf{8}}$$

$\Rightarrow$ AE = 8 $\times$ Sin 20° (By cross-multiplication)

$\Rightarrow$ AE = 8 $\times$ 0.3420201

$\Rightarrow$ AE = 2.736 km.

$$\frac{\cos 20^\circ}{1} = \frac{PE}{8}$$

$\Rightarrow$ PE = 8 $\times$ Cos 20° (By cross-multiplication)

$\Rightarrow$ PE = 8 $\times$ 0.9396926

$\Rightarrow$ PE = 7.518 km.

Notice that, since ACDE is a rectangle,

AE = CD and AC = ED (i.e. opposite sides are equal)

$\therefore$ BD = BC + CD = 3.830 + 2.736 = 6.566 km

and PD = PE + ED = 7.518 + 3.214 = 10.732 km.

Then $PB^2 = 6.566^2 + 10.732^2$

$\Rightarrow$ PB^2 = 43.112356 + 115.17582 = 158.28818

$\Rightarrow$ PB = $\sqrt{\mathbf{158.28818}}$

Þ **<u>PB</u>** = **12.581 km.** (i)

Angle BPD = Inverse Tan $\frac{\mathbf{BD}}{\mathbf{PD}}$

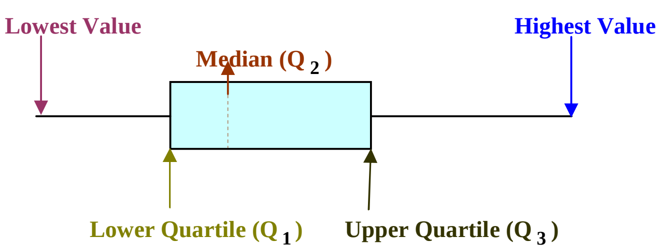
$\Rightarrow$ $\angle$ BPD= Inverse Tan $\frac{\mathbf{6.566}}{\mathbf{10.732}}$ = 31.5°

$\therefore$ $\angle$ NPB= 90° - 31.5° = 58.5°

Þ **<u>Bearing</u>** is **058.5°** (ii).

Box & whisker diagram

<u>BOX AND WHISKER DIAGRAM</u>



See **<u>Cumulative Frequency diagram</u>** (Ogive) below.

If we divide the **total frequency** into **quarters**, the **score** corresponding to the **lower quarter** is the **lower quartile**, the **score** corresponding to the **middle** is the **median** and the **score** corresponding to the **upper quarter** is the **upper quartile**.

The **interquartile range** is the **difference** between the **score readings** provided by the **upper** and **lower quartiles**.

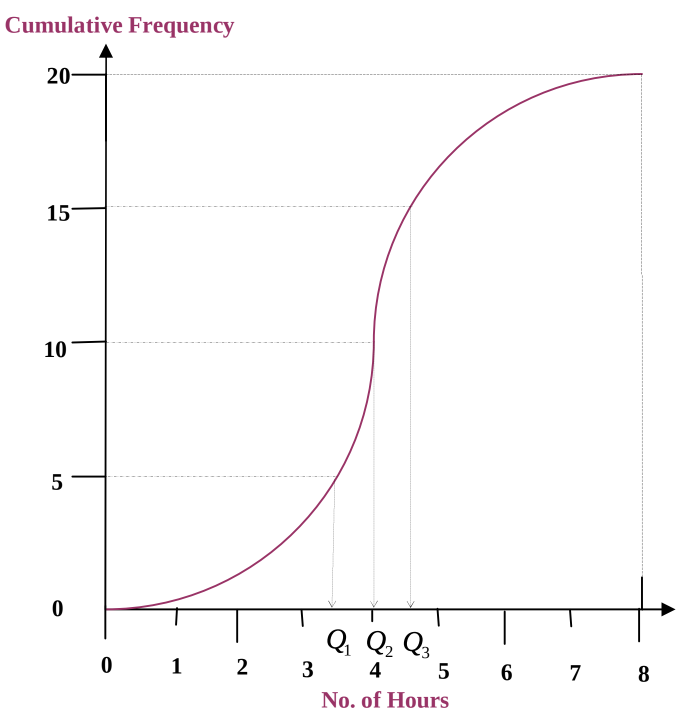
E.g. In a survey, **20** children were asked how many hours they spend on sports in each week.
The results of the survey were as follows:

4	**5**	**1**	**4**	**5**
3	**4**	**2**	**6**	**4**
4	**4**	**5**	**5**	**7**
5	**4**	**5**	**8**	**5**

Cumulative Frequency Table from given data

Number of Hours (less than or equal to)	**Cumulative Frequency**
0	**0**
1	**1**
2	**2**
3	**3**
4	**10**
5	**17**
6	**18**
7	**19**
8	**20**

The **cumulative frequency diagram** (ogive) compiled from the above data looks like this:



If we refer these results to a **Box & Whisker Diagram**, we have:
Lowest value = **1**; Highest value = **8**; Q_1 = **3.4**; Q_2 = **4**; Q_3 = **4.6**.
This gives: **Median** no. of hours = **<u>4</u>**. **Interquartile range = 4.6 – 3.4 = <u>1.2</u>** hours.

Brackets

Brackets are used like a **pocket** to hold things safely together. The **contents** of brackets must be treated as a **single quantity** and **worked out on their own**:

Eg. **3(x + 1)** means **1 is added to x** and the **result multiplied by 3**.

Eg. **2(3x - 1)** means **1 is subtracted from 3 times x** and **the result** is **doubled**.

To **remove brackets**, **multiply each term inside** the brackets by the '**multiplier**' **outside** the **brackets**.

Eg. **2(3x - 1) = 6x – 2** and **-2(3x - 1) = -6x + 2**.
Note: **6x – 2** is a **sum** of **terms** and **2(3x - 1)** is a **product** of its **factors**.
Also: **-6x + 2** is a sum of terms and **-2(3x - 1)** is a product of its factors.

When **simplifying** algebraic expressions, it is necessary to remove any brackets as a first step:

E.g. Simplify **2(3x - 1) – 3(6x – 2)**.
2(3x - 1) – 3(6x – 2) = 6x – 2 – 18x + 6 = - 12x + 4 = 4(1 – 3x).

Calculus

(Latin: '*calculus*' means 'pebble'.) (GCSE Additional Pure and Advanced Subsidiary Pure)
The **Calculus** is one of the most useful disciplines in mathematics. It deals with **changing quantities**.

The **Calculus** has **two** main **branches**:

(i) **Differential Calculus** (i.e. **differentiation**)
(ii) **Integral Calculus** (i.e. **integration**).

(i) The central problem of **differential calculus** is to find the **rate** at which a **known**, but **varying quantity**, **changes**. Generally, the **notation** used for the **derivative** of **y** with respect to **x** is the Leibnizian $\frac{dy}{dx}$.

$\frac{dy}{dx}$ gives the **gradient** of the **tangent** to the curve of **y** at *any point* on the curve.

E.g. If $y = 2x^2 + x - 1$, find the **gradient** of the curve at the point where **x = 1**, and, *hence*, find the **equation** of the **tangent** to the curve at the point **(1, 2)**.

$$y = 2x^2 + x - 1$$
$$\Rightarrow \quad \frac{dy}{dx} = 4x + 1$$
$$x = 1 \quad \Rightarrow \quad \frac{dy}{dx} = 4(1) + 1 = 5.$$

\ the **gradient** of the curve at the point where **x = 1** is **5**.

Q the **gradient** of the **tangent** is also **5**, we have:

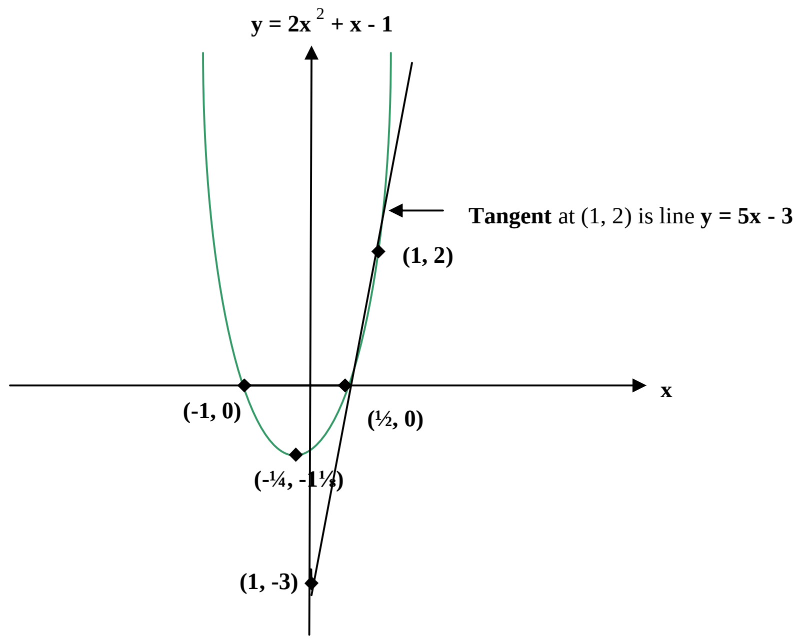
y = mx + c *(Remember the tangent is a straight line.)*
(1, 2) and **m = 5** $\Rightarrow$ **2 = 5(1) + c**
$\Rightarrow$ **2 = 5 + c**
$\Rightarrow$ **-3 = c**

\ **y = 5x – 3** is the equation of the **tangent** to the curve,

$y = 2x^2 + x – 1$, at the point **(1, 2)**.

`

See the diagram below:



(ii) **Integral calculus** has the **reverse** problem.
It tries to find a quantity when its **rate** of **change** is known.
Generally, the **notation** used for the **integral** is the Leibnizian
ò **y dx**.

ò **y dx** gives *any area* bounded by the curve of **y** and the **x-axis**.

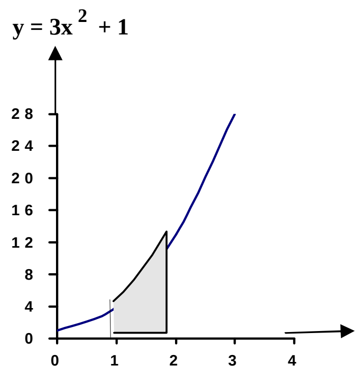
The integral ò **y dx** gives the **general area** between the **curve** of **y** and the **x-axis** – this is an **indefinite** integral.
When **limits** are known, an **actual area** may be calculated – this is a **definite** integral.

In a definite integral, there is **no need** for **c**, the constant of integration.

E.g. Find $\int_1^2 (3x^2 + 1)dx$.

(This means: *find the* ***area*** *bounded by the curve, y = 3x* ^{2} *+ 1, the* ***x-axis*** *and the lines x = 1 and x = 2*.)

The diagram looks like this, with the required area shaded:



$$\int_1^2 (3x^2 + 1)dx = \left[x^3 + x\right]_1^2 .$$

Notice the notation: [ ] is used once the integration has been done and the **limits** for **x** are moved to the right side.

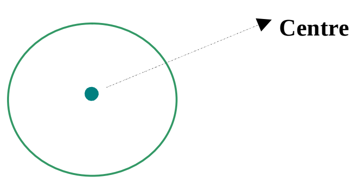
Now, **evaluate** this integral using the limits, **1** and **2**:

$\left[2^3 + 2\right] - \left[1^3 + 1\right]$ = **10 - 2 = 8** sq. units for shaded region.

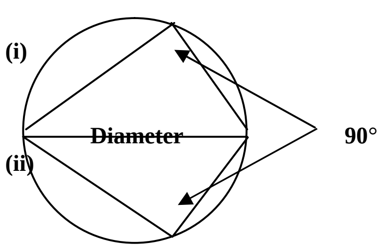
N.B. There is an **inverse relationship** between **differentiation** and **integration**. This **inverse relationship** is **'The Fundamental Theorem of the Calculus'**: it means that one process undoes the other, as, for example, **addition** of **2** undoes **subtraction** of **2**, or **multiplication** by **3** undoes **division** by **3**.

Circle

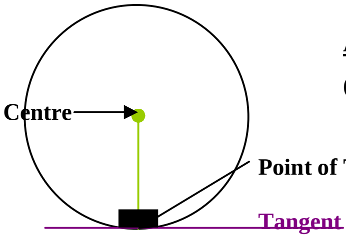
A **circle** is the locus of **all points equidistant** from a **fixed** point called the **centre**.



Circle Theorems



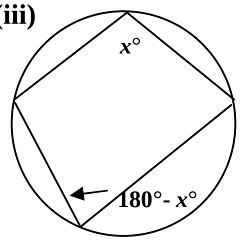
The angle in a semi-circle is a right angle.
(**N.B.** The converse is true also:
if a chord subtends an angle of 90° at the circumference, that chord must be a diameter.)



A radius is perpendicular to a tangent at the point of tangency.

(Conversely, a line perpendicular to a tangent at the point of tangency must go through the centre.)

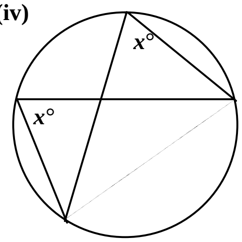
(iii)



Opposite angles in a cyclic quadrilateral are supplementary.

(Conversely, if opposite angles in a quadrilateral add up to 180°, that quadrilateral is cyclic, i.e. it could be circumscribed by a circle.)

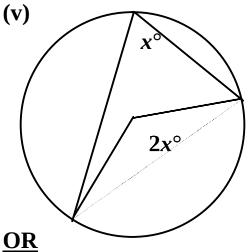
(iv)



Angles in the same segment are equal.

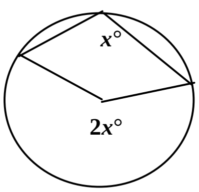
(The converse of this theorem, also, could be used to prove that the 4 points form a cyclic quadrilateral.)

(v)

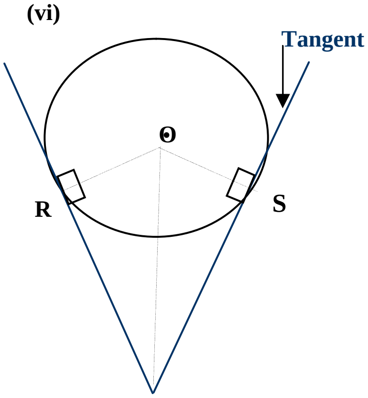


The angle subtended at the centre by an arc is twice any angle at the circumference, subtended by the same arc.

OR



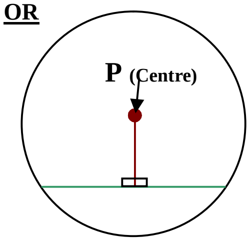
(vi)



If, from a point outside a circle, two tangents are drawn, these tangents are equal in length, i.e**. RP = SP** on the diagram.

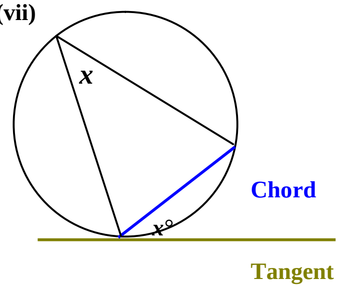
Notice that **ROSP** is, therefore, a **kite**, in which **OP** is the **line of symmetry**:
Q RO = SO (both radii),
∠ ORP = ∠OSP = 90° (angles between radii)
and OP is common to both triangles ORP and OSP.

OR



A line through the centre, perpendicular to a chord, bisects that chord.
(Conversely, a line that bisects a chord, must go through the centre.)

(vii)

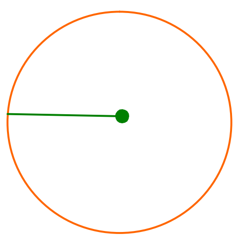


The angle between a tangent and a chord is equal to the angle in the alternate segment.

Circumference

The **circumference** is the **perimeter** of a circle;
π × diameter or **2 × π × radius** give the circumference.

Circumference of a **circle**, radius 0.5cm, = 2 × π × 0.5 = π cm.



Class boundary

The class interval (**10 – 20**) cm, rounded to the nearest cm, has **class boundaries 9.5 cm** and **20.5 cm** (lower and upper respectively).

Class frequency

When a large amount of numerical data is being handled, it is convenient to group the scores into **classes**, and obtain a **class frequency** for each class.

E.g. The class interval (**10 – 20**) cm could have **any** frequency.

Class interval

Objects, whose lengths are between **10cm** and **20cm**, are said to lie within the **class interval** (**10 – 20**) cm.

Class limit

In the **class interval** (**10 – 20**) cm, the **class limits** are **10 cm** (lower) and **20 cm** (upper).

Class width

The **class interval** (**10 – 20**) cm, rounded to the **nearest cm**, has **class width 11cm** (20.5 – 9.5, that is the **difference** between the upper and lower **class boundaries**). This is particularly important when drawing a **histogram**; the **bar** in this case must have width **11 cm**, ***not*** **10 cm**.

Compound Interest

For **simple interest**, the **total** number of years is used at once. However, for **compound interest**, we must take **one year** at a time, remembering to **add** the **interest** accumulated in the year to the starting **principal**, thereby **increasing** the **principal** at the end of each year.You may use the Simple Interest Formula for compound interest, if you take **Y = 1** and work out the interest for each year, *remembering to add in the interest to the principal as you go along.*

We shall now take the example considered under simple interest and see the difference when we use compound interest instead.

E.g. £200 is invested for 3 years at 10% per annum *compound* interest. Find the amount at the end of the 3 years.

<u>1st Year</u>:

Principal	£200
+ Interest = 10% of £200	£ 20
Principal + Interest	£220

<u>2nd Year</u>:

Principal	£220
+ Interest = 10% of £220	£ 22
Principal + Interest	£242

3rd Year:

Principal	£242
+ Interest = 10% of £242	£ 24.20
Principal + Interest	£266.20.

The amount at the end of 3 years is, therefore, **£266.20**, compared with **£260** when simple interest is used.

Using the method above, the work on compound interest problems can be laborious and time-consuming, particularly when the period of time in the problem is lengthy.

Again, there is a formula that can be used very conveniently:

Compound Interest Formula:

A	=	**Amount of money after Y years**
P	=	**Principal**
R	=	**Rate % per annum**
Y	=	**No. of Years**
I	=	**Interest**

$$A = P\left(1 + \frac{R}{100}\right)^{Y}$$

Next we shall demonstrate the use of the Compound Interest Formula in answering the question posed in the example above.

$$A = P\left(1 + \frac{R}{100}\right)^{Y}$$

A	=	**Amount after 3 years**
P	=	**£200**
R	=	**10 %** per annum
Y	=	**3**

$$A = 200\left(1 + \frac{10}{100}\right)^{3}$$

Þ $A = 200$ ´ 1.1^{3}

Þ $A = 200$ ´ 1.331

\ **A** = **£266.20** (as before).

Conditional probability

This means the probability that event **A** occurs, given that event **B** *has occurred already*; the **notation** used for this event is **P(A/B)**.

In this case the multiplication law gives:

$$\mathbf{P(A \cap B) = P(A/B) \times P(B)}$$

$$\Rightarrow \quad \mathbf{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$ (By rearrangement of the above.)

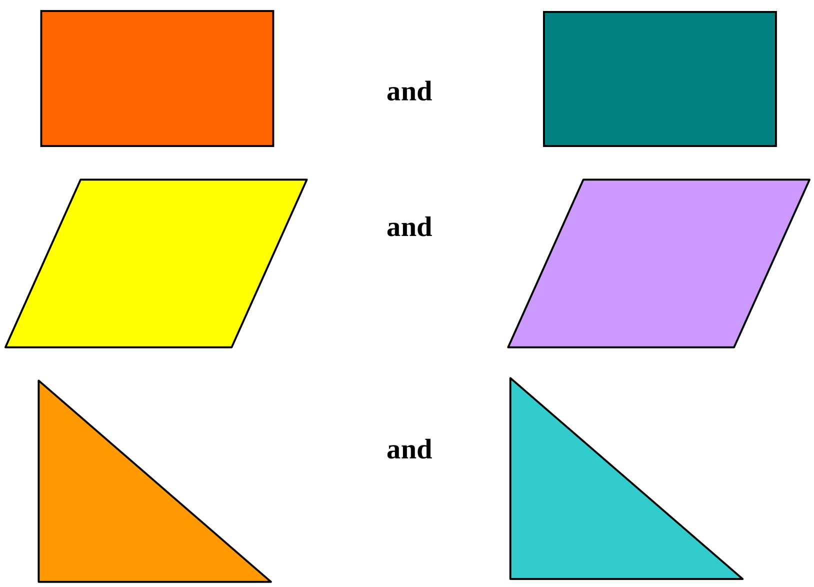
The **events** are said to be **independent** when the probability of either event occurring is *unaffected* by the probability of the *other* event having occurred.

In this case the multiplication law gives:

$$\mathbf{P(A \cap B) = P(A) \times P(B)}.$$

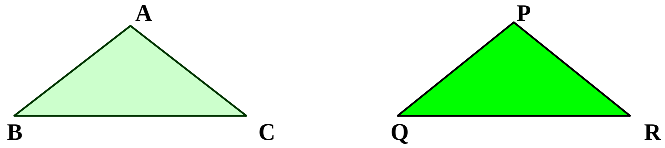
Congruent

Congruent shapes are equal in ***every*** respect; **corresponding sides** *and* **corresponding angles** are **equal** to each other.



Congruent triangles

Triangles that are **equal** in **every** respect are congruent.
In the diagram below, **ABC** and **PQR** are congruent triangles:



Ð **A** = Ð **P**,
Ð **B** = Ð **Q**,
Ð **C** = Ð **R**
and
BC = **QR**,
AC = **PR**,
AB = **PQ**.

Notice that the **angles** that are **equal** are **opposite** to the **sides** that are **equal**.
The **notation** used for congruence is º.

Therefore, **ABC** º **PQR** says that **ABC** and **PQR** are **congruent** triangles.

If we are required to **prove** that **two triangles** are **congruent**, it is sufficient to find that they satisfy **any one** of the following conditions:

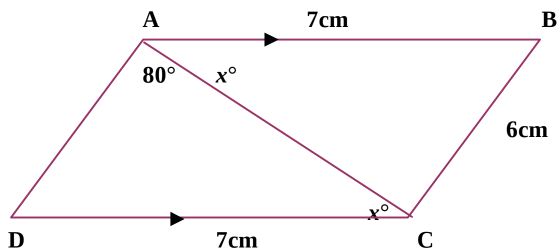
(i) **Two sides** and the **included angle** in one triangle equal to two sides and the included angle in the other. (**SAS**)

(ii) **One side** and **two angles** in one triangle equal to one side and two ***similarly-positioned angles*** in the other. (**ASA**)

(iii) **Three sides** of one triangle **equal** to the **three** sides in the other. (**SSS**)

(iv) **Right angle**, **hypotenuse** and **one other side** in each triangle equal. (**RHS**)

E.g.



In the diagram above, **AB** and **CD** are parallel and each has length **7cm**.
If **BC** = **6cm** and Ð**DAC** = **80°**, prove that the triangles **ADC** and **ABC** are congruent.

Q **AB** is parallel to **DC**, Ð **BAC** = Ð **ACD** (alternate angles**)**.

Also, **Q** **AB** = **BC** (given) and **AC** is common to both triangles **ADC** and **ABC**, we have **two sides** and the **included angle** in **ADC** equal to two sides and the included angle in **ABC**.

\ **ADC** º **ABC** (**Q** Condition **SAS** is satisfied).

Q.E.D.

Connected particles (Mechanics – GCSE Additional and Advanced Subsidiary)

In these problems, the **strings** connecting two particles are considered to be **light** and **inextensible**:

Since the string is *light*, its **weight** can be **ignored**.

Also, since it is *inextensible*, both particles have the **same speed** and **acceleration** while the string is kept *taut*.

By Newton's Third Law, the **tension** in the **string** acting on **both particles** is **equal in magnitude** and **opposite in direction**.

N.B. A *smooth* surface offers **no resistance** to the motion of a body across it.

Example 1:

Two particles, **P** and **R** are at rest on a smooth horizontal surface.
P has mass **3kg** and **R** has mass **4kg**.

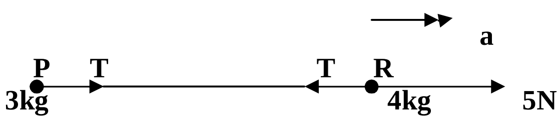
P is connected to **R** by a light inextensible string, which is taut.

A force of **5N** is exerted on particle **R** in the direction **PR**.

Find the acceleration of the two particles and the tension in the string.

Method:

The **diagram** looks like this:



Using **F = ma**:

Particle **P**:	**T**	=	**3a**	… (i)
Particle **R**:	**5 – T**	=	**4a**	… (ii)
(i) + (ii)	**5**	=	**7a**	
\	$\frac{5}{7}$	=	**a**.	
Substitute in (i): **T**		=	$2\frac{1}{7}$.	

∴ the **acceleration** of the particles is $\frac{5}{7}$ **m/s**^{2}
and the **tension** in the string is $2\frac{1}{7}$ **N**.

Example 2:

Two particles, **P** and **R** are connected by a light inextensible string passing over a smooth fixed pulley.

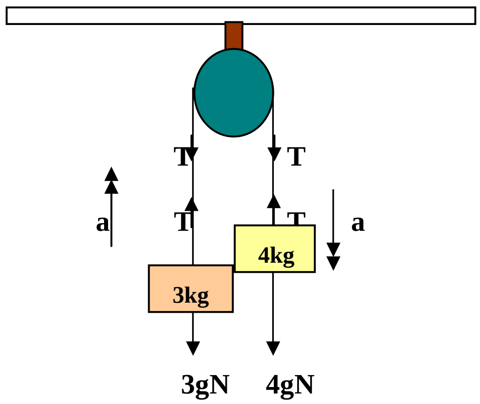
P has mass **3kg** and **R** has mass **4kg**.

The particles hang freely and are released from rest.

Find the **acceleration** of the two particles and the **tension** in the string.
(Take g = 9.8.)

Method:

The **diagram** looks like this:



Using **F = ma**:

Particle **P**: $\mathbf{T - 3g} = \mathbf{3a}$ … (i)

Particle **R**: $\mathbf{4g - T} = \mathbf{4a}$ … (ii)

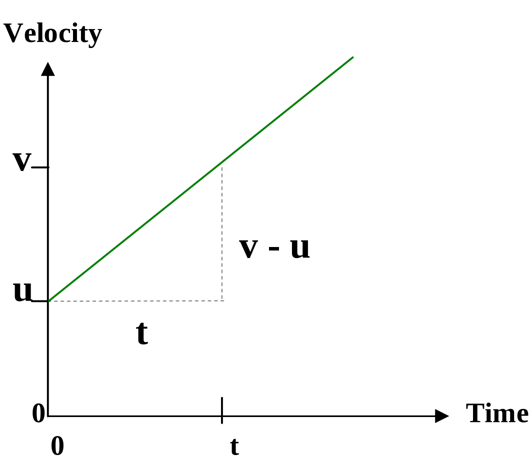
(i) + (ii) $\mathbf{g} = \mathbf{7a}$

\ $\frac{1}{7}\mathbf{g} = \mathbf{a}$.

Substitute in (i): $\mathbf{T} = \mathbf{3g} + \frac{3}{7}\mathbf{g}$.

∴ the **acceleration** of the particles is $\mathbf{1.4\ m/s^2}$
and the **tension** in the string is **33.6 N**.

Constant acceleration formulae (Mechanics - GCSE Additional and Advanced Subsidiary)



Let **u** be the **initial velocity** and **v** be the **velocity at time t**.

Since the **gradient** of a **velocity/time graph** gives **acceleration**, we have:

$$\mathbf{a} = \frac{\mathbf{v - u}}{\mathbf{t}}$$

Þ $\mathbf{at} = \mathbf{v - u}$ **(By cross-multiplication)**

or $\mathbf{v} = \mathbf{u + at}$ … **Formula (i)**

Since **area** under a **velocity/time graph** gives **displacement**, we have:

$\mathbf{s} = \frac{1}{2}(\mathbf{u + v})\mathbf{t}$ **(Area of Trapezium)**

Þ $\mathbf{s} = \frac{1}{2}(\mathbf{u + u + at})\mathbf{t}$ **(From (i) above)**

Þ $\mathbf{s} = \frac{1}{2}(\mathbf{2u + at})\mathbf{t}$

or $\mathbf{s} = \mathbf{ut} + \frac{1}{2}\mathbf{at}^2$ … **Formula (ii)**

$\mathbf{v}^2 = (\mathbf{u + at})^2$ **(From (i) above)**

Þ $\mathbf{v}^2 = \mathbf{u}^2 + \mathbf{2uat} + \mathbf{a}^2\mathbf{t}^2$

Þ $\mathbf{v}^2 = \mathbf{u}^2 + \mathbf{2a}(\mathbf{ut} + \frac{1}{2}\mathbf{at}^2)$

Þ $\mathbf{v}^2 = \mathbf{u}^2 + \mathbf{2as}$ … **Formula (iii)**
(From (ii) above)

(i), **(ii)** and **(iii)** above are the **CONSTANT ACCELERATION FORMULAE**.

Correlation

If, in a set of paired observations from two random variables, a **change** in **one** variable is **matched** by a **similar proportional change** in the **other** variable, then the technique used to measure the degree of association is called **correlation**.

Positive correlation: **both variables increase** together.

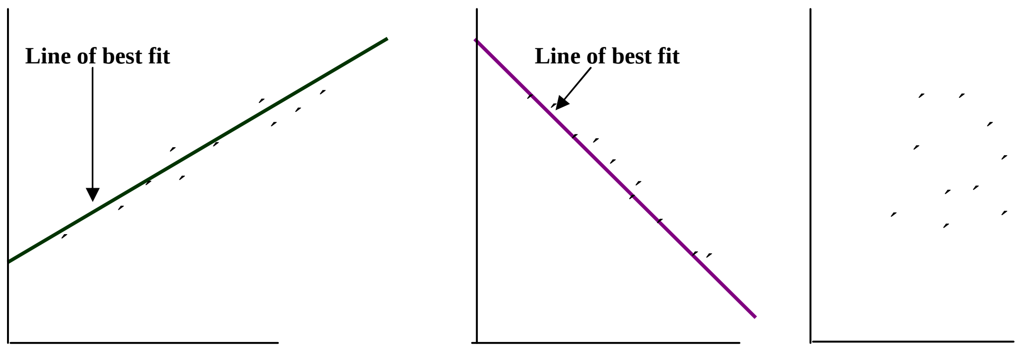
Negative correlation: as **one variable increases**, the **other decreases**.

A **scatter diagram** is used to determine whether there is a linear relationship between two variables – when there is evidence of correlation, a **line of best fit** can be drawn through the points – see the diagrams below:

Positive Correlation
- i.e. positive gradient

Negative Correlation
- i.e. negative gradient

No Correlation

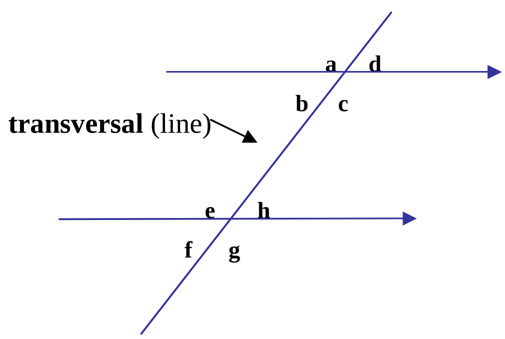


Corresponding angle

Corresponding angles are the angles in a **corresponding position** on the **same side** of a **transversal** drawn through a pair of **parallel** lines; these angles are **equal** to each other.

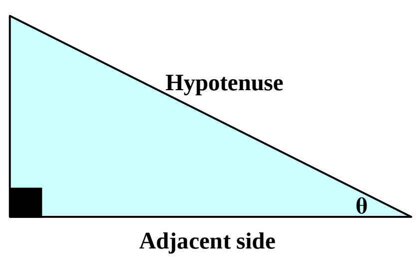
Corresponding angles in the **diagram below**:

a = e   **b = f**   **d = h**   **c = g**



Cos(ine)   **Cos**(ine) is the **trigonometric ratio** $\frac{adjacent}{hypotenuse}$ in a **right - angled triangle**.

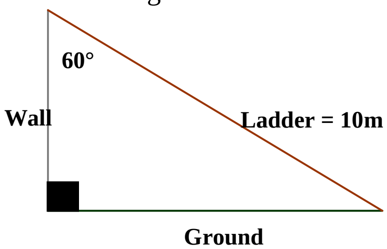
Cos θ = $\frac{adjacent}{hypotenuse}$ **-** see diagram below:



Worked Example (use of cosine):

A ladder, **10m** long, is placed against a wall at an angle of **60°** to the ground.
Find how far up the wall the ladder reaches

Draw a diagram:



$$\text{Cos } 60° = \frac{wall}{ladder}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{wall}{10}$$

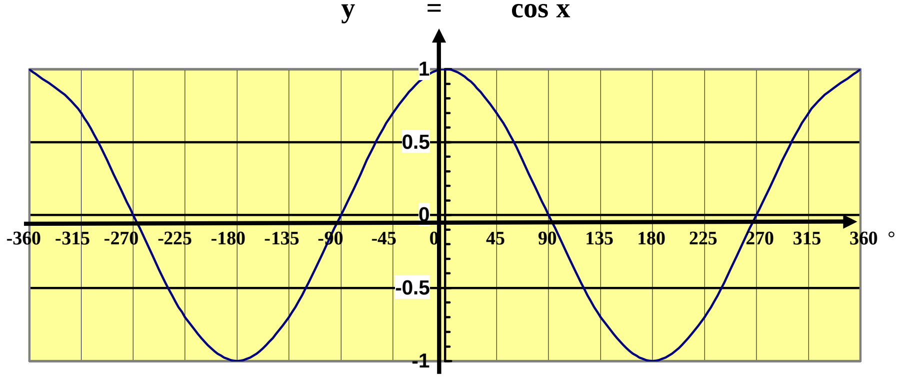
$$\Rightarrow \quad 2 \times \text{wall} = 10$$

\ **wall** = **5m**.

Cosine Graph:

Since **0°**, **360°**, **720°**, …all have the same **cosine**, as do **180°**, **540°**, … and so do **90°**, **450°**, … , the cosine graph starts to **repeat** after every **360°**; it is said to have **period 360°**.

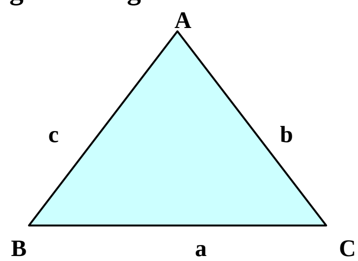
y = cos x



N.B. Notice how the graph of **y = cos x** oscillates about the ***x*-axis** between **y = 1** and **y = -1**.

Cosine rule

The **cosine rule** (along with the **sine rule** - see **sin rule**) is used to solve **non-right-angled triangles**.

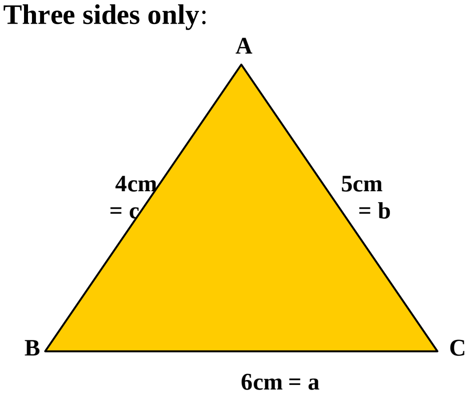


The **Cosine Rule** states:

$$\mathbf{a^2 = b^2 + c^2 - 2bcCos\ A} \quad \text{or}$$
$$\mathbf{b^2 = a^2 + c^2 - 2acCos\ B} \quad \text{or}$$
$$\mathbf{c^2 = a^2 + b^2 - 2abCos\ C}.$$

The **cosine rule** *must* be used when the following information is given:

- **Three sides only**:

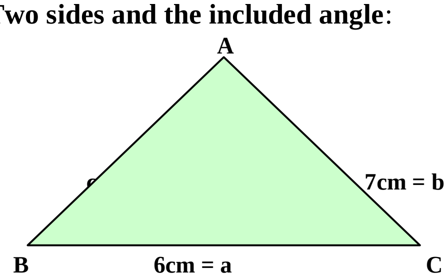


Using the **cosine rule** to find *x*°, we have:

$$\mathbf{a^2 = b^2 + c^2 - 2bcCos\ A}$$
$$\Rightarrow \mathbf{36 = 25 + 16 - 40Cos\ A}$$
$$\Rightarrow \mathbf{CosA = \frac{25 + 16 - 36}{40} = \frac{5}{40} = \frac{1}{8}}$$
$$\Rightarrow \mathbf{A = 82.8°}.$$

To find **another angle**, use the **sine rule** and the triangle is then solved completely.

- **Two sides and the included angle**:



Using the **cosine rule** to find $x°$, we have:

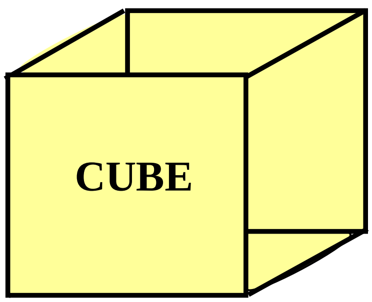
$$c^2 = a^2 + b^2 - 2abCos\ C$$
$$\Rightarrow c^2 = 36 + 49 - 84Cos\ 65°$$
$$\Rightarrow c^2 = 49.5$$
$$\backslash \quad c = 7.04cm.$$

To find **another angle**, use the **sine rule** and the triangle is then solved completely.

Cubic number

Cubic numbers give the **volumes of cubes of edge 1, 2, 3,**. . .

We have: $1^3 = 1 \times 1 \times 1 = 1$
$2^3 = 2 \times 2 \times 2 = 8$
$3^3 = 3 \times 3 \times 3 = 27$ and so on.



Cubic numbers are often referred to as **cubes.**
Cubic numbers = {1, 8, 27, 64, 125, . . . }.

Cumulative frequency

Score frequencies from a **frequency table** are added **cumulatively** to obtain a **cumulative frequency table**.

E.g. The heights recorded in the frequency table below have been rounded off to the nearest cm.
Compile a cumulative frequency table from the data.

Frequency Table

Height (cm)	Frequency
145-149	5
150-154	8
155-159	15
160-164	35
165-169	20
170-174	10
175-179	7
TOTAL	100

Cumulative Frequency Table

Height (cm) – less than	Frequency	Cumulative Frequency
144.5	0	0
149.5	5	5
154.5	8	13
159.5	15	28
164.5	35	63
169.5	20	83
174.5	10	93
179.5	7	100

Cumulative frequency diagram (ogive)

E.g. The heights recorded in the frequency table below have been rounded off to the **nearest cm**.

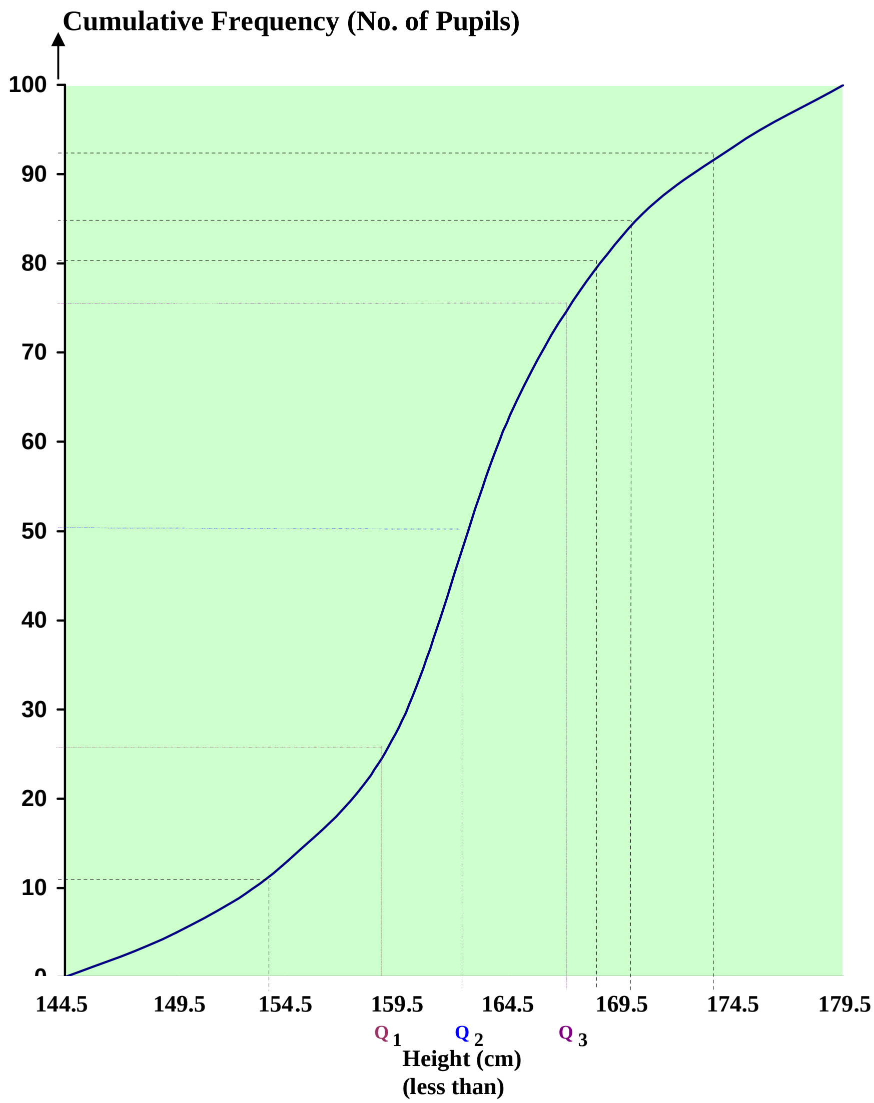
Compile a cumulative frequency table from the data and draw a cumulative frequency graph.

Frequency Table

Height (cm)	Frequency
145-149	5
150-154	8
155-159	15
160-164	35
165-169	20
170-174	10
175-179	7
TOTAL	100

Cumulative Frequency Table

Height (cm) – less than	Frequency	Cumulative Frequency
144.5	0	0
149.5	5	5
154.5	8	13
159.5	15	28
164.5	35	63
169.5	20	83
174.5	10	93
179.5	7	100



The main types of information that can be gained from the ogive are:

(a) the **interquartile range**

and

(b) the **median**.

If we divide the **total frequency** into **quarters**, the **score** corresponding to the **lower quarter** is the **lower quartile**, the **score** corresponding to the **middle** is the **median** and the **score** corresponding to the **upper quarter** is the **upper quartile**.
The **interquartile range** is the **difference** between the **score readings** provided by the **upper** and **lower quartiles**.

From our diagram above, the readings are:

(a) **Interquartile Range**:

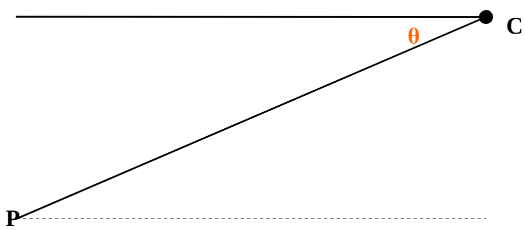
$$Q_3 - Q_1 = 167.25 - 158.7 = \underline{8.55\text{cm}}.$$

(b) **Median Height**:

$$Q_2 = \underline{162.75\text{cm}}.$$

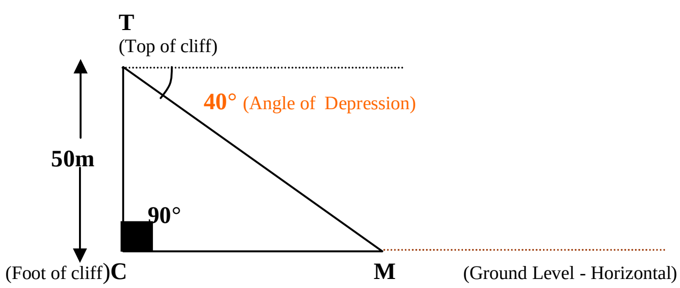
Depression – angle of

The **angle of depression** could be described as the angle of "**drop**" from a point on a **higher horizontal** level to a point on a **lower horizontal** level.



θ is angle of depression of P from C

E.g. If the angle of depression from the top **T** of a cliff, **50m** high, to a marker **M** on the ground is **40°**, the **diagram** looks like this:



With the aid of **trigonometry**, it is possible to find the distance **CM**.

$\angle$ MTC = 50° (i.e. 90° – 40°)

$\therefore \dfrac{\tan 50°}{1} = \dfrac{MC}{50}$

$\Rightarrow MC \times 1 = 50 \times \tan 50°$ (By cross-multiplication)

$\Rightarrow$ **MC = 50 × 1.1917536 = 59.588m**

Differentiation (GCSE Additional Pure and Advanced Subsidiary Pure)

Differentiating a function is equivalent to finding the **gradient** of the curve.

The **gradient** of a curve at **any point** is the same as the **gradient** of the **tangent** to the curve at *that* point.

$\dfrac{dy}{dx}$ (or **f′(x)**, if you prefer), then, gives a **general expression** for the **gradient** of a curve at **any point** on the curve.

All that is required to find the **actual gradient** at a **particular point** is to substitute the value of **x** at *that point* into the **general** expression for the **gradient**, i.e. the derivative, $\frac{dy}{dx}$.

The **technique** for **differentiating** a function is as follows:

1. **Premultiply** by **index**.
2. **Subtract 1** from index to give **new index**.
3. The **derivative** of a **constant** is **0**.
4. **Coefficients** are **not affected** by differentiation.

E.g. If $y = 2x^2 + x - 1$, find the **gradient** of the curve at the point where $x = 1$, and, *hence*, find the **equation** of the **tangent** to the curve at the point **(1, 2)**.

$$y = 2x^2 + x - 1$$

$$\Rightarrow \quad \frac{dy}{dx} = 4x + 1$$

$$x = 1 \quad \Rightarrow \quad \frac{dy}{dx} = 4(1) + 1 = 5.$$

$\therefore$ the **gradient** of the curve at the point where $x = 1$ is **5**.

$\because$ the **gradient** of the **tangent** is also **5**, we have:

$y = mx + c$ *(Remember the tangent is a straight line.)*

(1, 2) and $m = 5 \Rightarrow 2 = 5(1) + c$

$\Rightarrow \quad 2 = 5 + c$

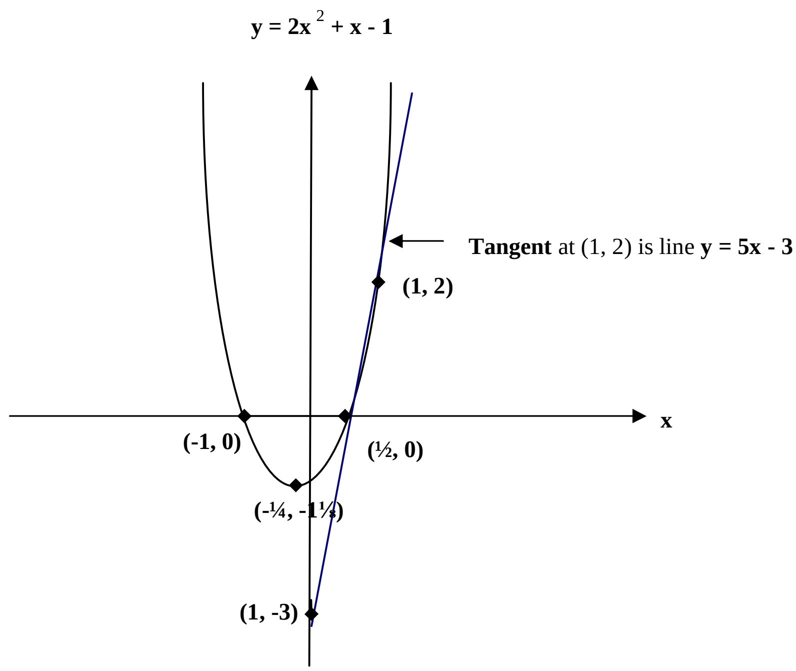
$\Rightarrow \quad -3 = c$.

$\therefore$ $y = 5x - 3$ is the equation of the **tangent** to the curve,

$y = 2x^2 + x - 1$, at the point **(1, 2)**.

`

See the diagram on the next page:

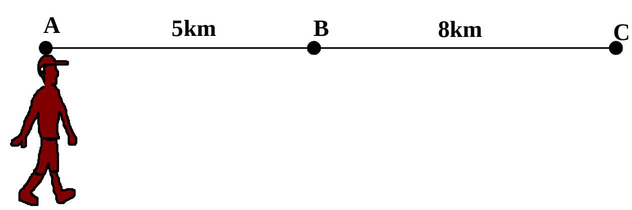


Displacement

A **displacement** is a **translation** of a point from one position to another through $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$. For positive values of **x** and **y** this is equivalent to $\begin{pmatrix} \textbf{Easting} \\ \textbf{Northing} \end{pmatrix}$; where the point is moved **x** units to the **east** and **y** units to the **north** – a **minus easting** can be thought of as a **westing** and a **minus** northing a **southing**.
A displacement is a vector quantity, generally in metres, **m**.
It is the distance between the starting point and the finishing point, *not necessarily* the **total** distance travelled.

E.g. Jonathan sets out walking from his home at **A**. He intends to stay for the night at his brother's house at **B**. He has to go on to **C** first to do some business before finally stopping at **B**.

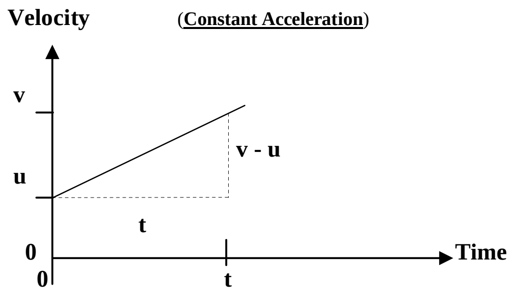
The diagram below aims to show the **difference** between **displacement** and **distance**:



Distance walked: 5km + 8km + 8km = **21km**.
Displacement of **B** from **A**: **5km**.

The **area** under a **velocity**/**time** graph gives the **displacement**.

E.g.



Let **u** be the **initial velocity** and **v** be the **velocity at time t**.

Since **area** under a **velocity/time graph** gives **displacement**:

$$\mathbf{s} \quad = \quad \frac{1}{2}(\mathbf{u}+\mathbf{v})\mathbf{t} \qquad \textbf{(Area of Trapezium)}.$$

- The **gradient** of a **displacement**/**time** graph gives the **velocity**.
  When the acceleration is **variable**, the **displacement**/**time** graph is **curved**.
  In this case, the gradient may be **approximated** by drawing a **tangent** to the curve at the required **time point** and calculating its gradient.

- The accurate way to find the **velocity** is to **differentiate** **displacement** with respect to **time**
  i.e. **v** (velocity) = $\frac{ds}{dt}$.

N.B. **Differentiation** is not always an option as it can be used only when an equation showing **displacement** in terms of **time** is known.

E.g. If the **displacement**, **s**, of a moving body after time **t** seconds is given by the equation:

$$\mathbf{s} \quad = \quad \frac{2}{3}t^3 - t^2 - 4t + 6,$$

find the velocity, **v**, after **3** seconds.

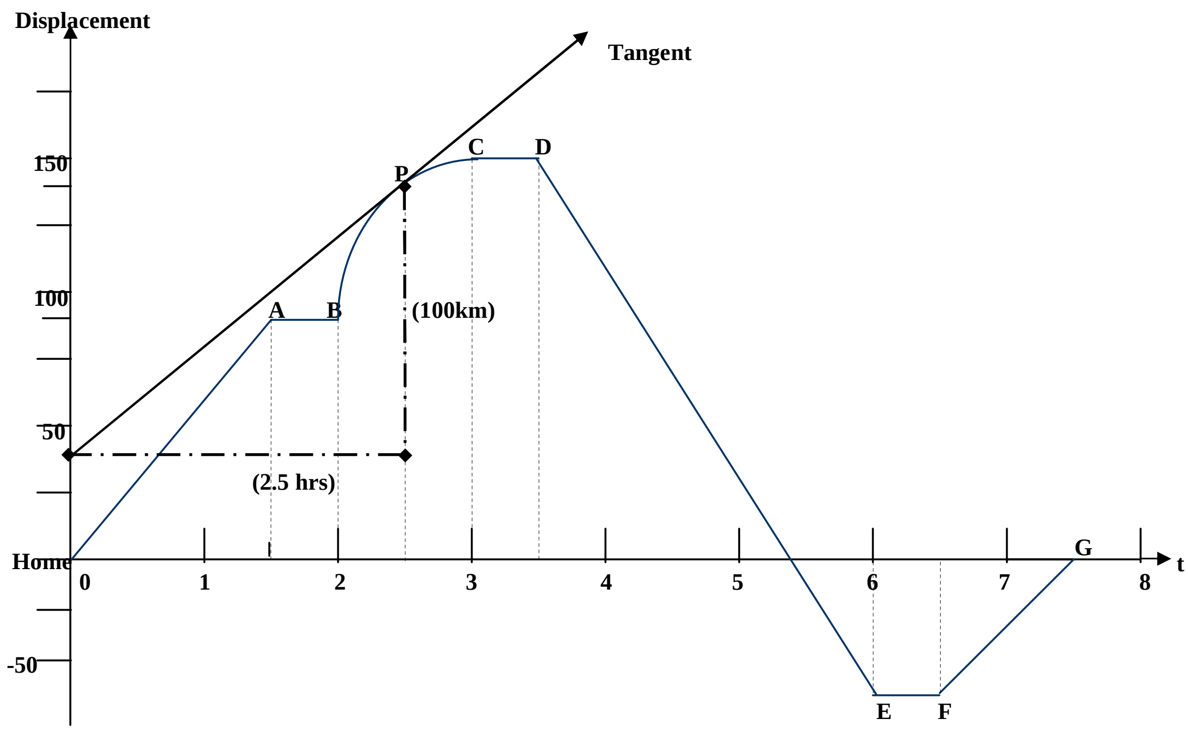
$$\mathbf{v} \quad = \quad \frac{ds}{dt} \quad = \quad 2t^2 - 2t - 4$$

$$\mathbf{t = 3} \quad Þ \quad \mathbf{v} \quad = \quad 2(3)^2 - 2(3) - 4$$

$$Þ \quad \mathbf{v} \quad = \quad \mathbf{8m/s} \quad \text{after } \mathbf{3}\text{seconds.}$$

<u>Displacement/Time</u> (or <u>Distance/Time</u>) <u>Graphs</u>

The following **graph** represents the journey of a motorist from home and back on a particular day.



The graph is interpreted as follows**:**

OA: Since this is a **straight** line, the **speed** is **constant**:

$$\frac{\textbf{distance}}{\textbf{time}} \quad Þ \quad \frac{\textbf{90}}{\textbf{1.5}} \quad = \quad \underline{\textbf{60km/h}}.$$

Notice that $\frac{\textbf{90}}{\textbf{1.5}}$ gives the **gradient** of the line **OA**,

i.e. $\frac{\textbf{displacement}}{\textbf{time}} \quad = \quad$ **velocity**.

This is *always* the case. Since **velocity** is the **rate** of **change** of **displacement** with **time**, the **gradient** of a **displacement/time** graph at any point gives the **velocity**.

AB: Since this is a **horizontal** line, the motorist is at **rest** for half an hour.

Observe that the **gradient** of this line is **zero**:

i.e. $\frac{\textbf{displacement}}{\textbf{time}} = \frac{\mathbf{0}}{\mathbf{0.5}} = \mathbf{0} =$ **velocity**.

Again this is *always* true. A **zero gradient** on a **displacement/time graph** indicates a **velocity** of **zero**, i.e. the **moving** ‘**object**’ is at rest.

BC: Since **BC** is a **curve**, the **velocity** varies, (i.e. it is not **constant**) for this part of the journey.
However, the **velocity** at any time **t** on **BC** can be calculated by drawing a **tangent** to the curve at that **point** for **t**, and finding its **gradient**.

N.B. The **gradient** of a **curve** at **any point** is the **same** as the **gradient** of the **tangent** to the **curve** at *that* point.

Displacement, Velocity & Acceleration using the calculus

(Mechanics - GCSE Additional and Advanced Subsidiary)

Since the **gradient** of a **displacement/time graph** gives the **velocity**, we **differentiate**,

i.e. $\mathbf{v} = \frac{\mathbf{ds}}{\mathbf{dt}}$.

Again, since the **gradient** of a **velocity/time graph** gives the **acceleration**, we **differentiate**,

i.e. $\mathbf{a} = \frac{\mathbf{dv}}{\mathbf{dt}}$.

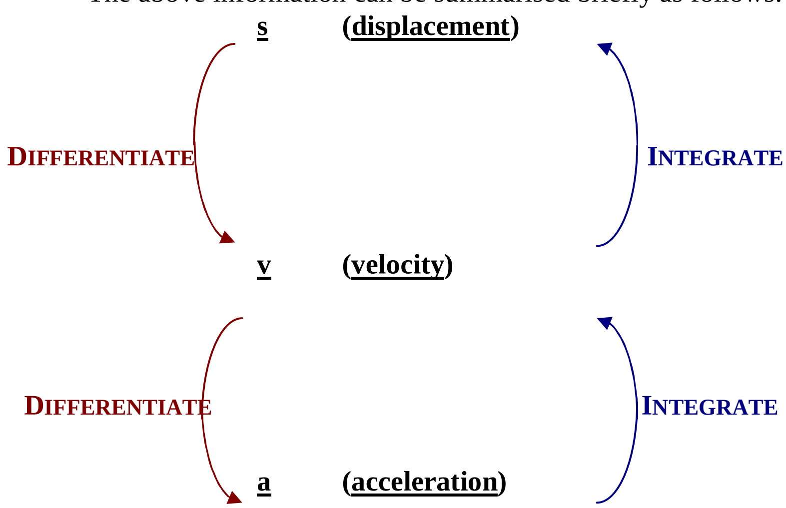
If we wish to go ‘**backwards**’ from **acceleration** to **velocity**, we **integrate**,

i.e. $\mathbf{v} = \int \mathbf{a}\ \mathbf{dt}$.

Going ‘**backwards’** again from **velocity** to **displacement**, we **integrate**,

i.e. $\mathbf{s} = \int \mathbf{v}\ \mathbf{dt}$.

The above information can be summarised briefly as follows:



E.g. The velocity, **v**, of a moving body is ($\mathbf{2t^2 + 1}$)m/s, after a time, **t** seconds. Find:

(i) the **displacement** after **2** seconds.

(ii) the **initial velocity**.

(iii) the **acceleration** after **1** second.

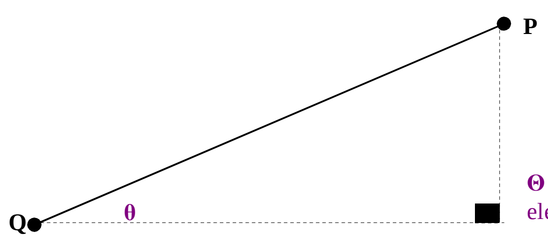
(i)
$$s = \int v\,dt + c$$
$$\Rightarrow \quad s = \int (2t^2 + 1)\,dt + c$$
$$\therefore \quad s = \frac{2t^3}{3} + t + c$$
$$t = 0,\ s = 0 \quad \Rightarrow \quad 0 = \frac{2(0)^3}{3} + 0 + c$$
$$\Rightarrow \quad c = 0$$
$$\therefore \quad s = \frac{2t^3}{3} + t\ .$$
$$t = 2 \Rightarrow s = \frac{2(2)^3}{3} + 2$$
$$\Rightarrow \quad s = 7\frac{1}{3}\text{ m.}$$

(ii) The **initial velocity** is at **t = 0**.
$$\therefore \quad v = 2(0)^2 + 1 = 1\text{m/s.}$$

(iii)
$$a = \frac{dv}{dt} = 4t$$
$$t = 1 \Rightarrow a = 4\text{m/s}^2 \text{ after } 1 \text{ second.}$$

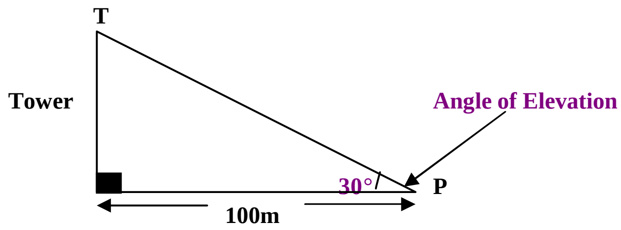
Elevation - angle of

The **angle of elevation** could be described as the angle of "**lift**" from a point on a **lower horizontal** level to a point **vertically above** that level.



Θ is the angle of elevation of **P** from **Q**.

E.g. If the angle of elevation from a point **P** on the ground to the top of a tower, **T**, whose base is **100m** from **P**, is **30°**, the **diagram** looks like this:



With the aid of **trigonometry**, it is possible to find the **height** of the tower:

$$\frac{\tan 30°}{1} = \frac{\text{Tower}}{100}$$

$\Rightarrow$ Tower $\times$ 1 = Tan 30° $\times$ 100 (By cross-multiplication)

Þ **Tower** = **0.5773502 ´ 100 = 57.735m.**

Enlargement

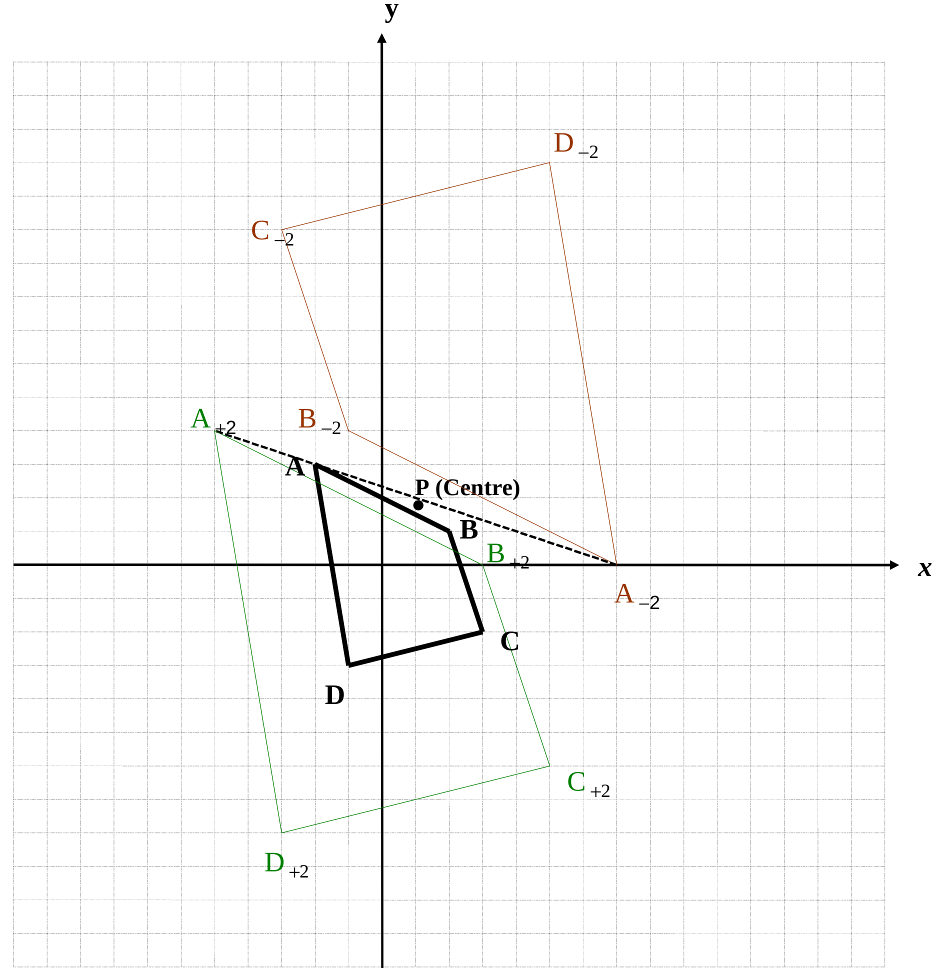
An **enlargement** is a **transformation** on a point using **scale factor k** with **centre** (**a, b**); it moves a point (**x, y**) to (x_1, y_1), where (x_1, y_1) is **k times** as far from the **centre** as (**x, y**).

E.g. The quadrilateral **ABCD** with **vertices** **(-2, 3)**, **(2, 1)**, **(3, -2)** and **(-1, -3)** respectively has

(i) $A_{+2}B_{+2}C_{+2}D_{+2}$ as its **image**, following an **enlargement** using **centre** (**1**, **2**) and **scale factor +2**.

(ii) $A_{-2}B_{-2}C_{-2}D_{-2}$ as its **image**, following an **enlargement** using **centre** (**1**, **2**) and **scale factor -2**.

(See the diagram on the next page.)



N.B. The **inverse** of an **enlargement** using **centre (x, y)** and **scale factor k** is an **enlargement**, **centre (x, y)**, **scale factor** $\frac{1}{k}$:

(i) **Scale factor** $+\frac{1}{2}$ (**centre P**) maps $\mathbf{A}_{+2}\mathbf{B}_{+2}\mathbf{C}_{+2}\mathbf{D}_{+2}$ **back** to **ABCD**

(ii) **Scale factor** $-\frac{1}{2}$ (**centre P**) maps $\mathbf{A}_{-2}\mathbf{B}_{-2}\mathbf{C}_{-2}\mathbf{D}_{-2}$ **back** to **ABCD**.

Equation The word **'equation'** means **'equality'**.

An equation is simply a statement that **two quantities** are **equal**,
i.e.: Left-hand Side = Right-hand Side

They would '**balance** ' if '**weighed on a pair of scales**'.

Example of arithmetic equations: $\mathbf{2 \times (9-1) = 4 \times 4}$

Left-hand side = **16** and Right -hand side = **16**.

Examples of algebraic equations:

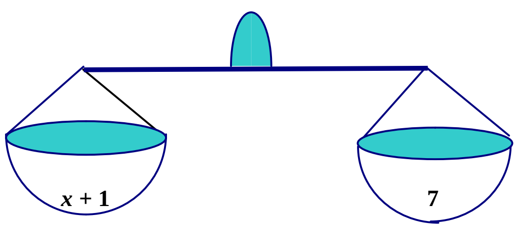
(i) $x + 1 = 7$

(ii) $2x + 3 = 15$

(iii) $2x + 1 = x + 5$.

(i) $x + 1 = 7$

If we put these quantities on the 'scales' they would **balance**:



Subtract 1 from each side:

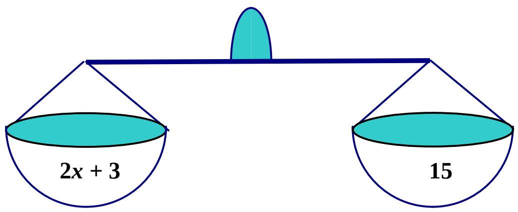
$$\begin{aligned} x + 1 &= 7 \\ -1 &\quad -1 \\ \hline x &= 6. \end{aligned}$$

We have just **solved the equation** $x + 1 = 7$, and found the **solution** to be:

$x = 6$. (Check: **6** + 1 = 7).

(ii) $\mathbf{2x + 3 = 15}$

Again, these quantities would **balance** on the scales:



$$2x + 3 = 15$$

***Subtract 3 from each side*:** $\underline{-3} \quad \underline{-3}$

$$2x = 12$$

***Divide each side by 2*:** $x = 6$.

We have **solved the equation** $\mathbf{2x + 3 = 15}$ and found **the solution** to be:

$x = \mathbf{6}$. (Check: $2 \times \mathbf{6} + 3 = 15$).

(iii) $\mathbf{2x + 1 = x + 5}$

***Subtract 1 from each side*:** $\underline{-1} \quad \underline{-1}$

$$2x = x + 4$$

***Subtract x from each side*:** $\underline{-x} \quad \underline{-x}$

$$x = 4.$$

$x = \mathbf{4}$ is the **solution**. (Check: $2 \times \mathbf{4} = \mathbf{4} + 4$).

Equation of motion

(Mechanics – GCSE Additional and Advanced Subsidiary)

Generally, a force of **F** newtons acting on a body of mass **m** kg produces an **acceleration** of **a** m/s^2, giving the **equation** of **motion**:

$$\mathbf{F = ma}.$$

E.g. 1. A **force** of **5N** acting on a body of **mass 10kg** has an **acceleration** of **0.5m/s**^{2}:

$$\mathbf{5 = 10a \Rightarrow a = 0.5}.$$

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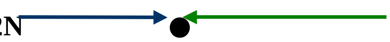
E.g. 2. The **resultant force** that would give a body of **mass 250g** an **acceleration** of **12m/s^2** is **3N**:

$$\begin{aligned} & \mathbf{F} && = && \mathbf{0.25(12)} \\ \Rightarrow\ & \mathbf{F} && = && \mathbf{3N}. \end{aligned}$$

E.g. 3. A body of **mass 2kg** rests on a smooth horizontal surface. Horizontal forces of **12N** and **7N** start to act on the particle in opposite directions.

Find the **acceleration** of the body.

12N



7N

2kg

$$\begin{aligned} & \mathbf{F} && = && \mathbf{ma} \\ \Rightarrow\ & \mathbf{12 - 7} && = && \mathbf{2a} \\ \Rightarrow\ & \mathbf{5} && = && \mathbf{2a} \\ \Rightarrow\ & \mathbf{2.5} && = && \mathbf{a} \end{aligned}$$

$\Rightarrow$ acceleration of **2.5m/s^2**.

F = ma can be applied easily in two or three dimensions using vectors.

Example:

Three forces, F = (2i – 3j + k) N, G = (i + 2j - 4k) N and H = (-4i – 2j + 5k) N, act on a particle of mass 10kg.
Find the resultant of these forces and, hence, the acceleration produced.

Resultant: **F + G + H** = **(**2i – 3j + k**)** + **(**i + 2j - 4k**)** + **(**-4i – 2j + 5k**)**
= **-i – 3j + 2k**.

$$\begin{aligned} & \mathbf{F} && = && \mathbf{ma} \\ Þ\ & \mathbf{-i - 3j + 2k} && = && \mathbf{10a} \\ Þ\ & \mathbf{-0.1i - 0.3j + 0.2k} && = && \mathbf{a} \end{aligned}$$

Þ acceleration of **(-0.1i – 0.3j + 0.2k) m/s^2**.

Equilibrium

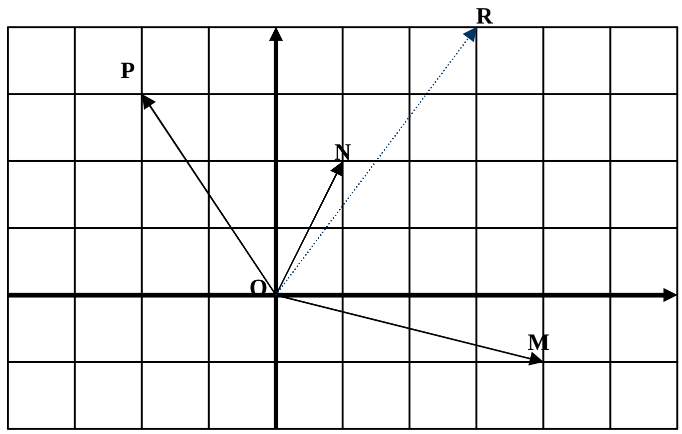
(Mechanics – GCSE Additional and Advanced Subsidiary)

If a body is **not moving**, then the **resultant force** in any direction must be **zero**.

Hence, if **R** is the **resultant** of three forces **P**, **N** and **M**, then **- R** added to **P**, **N** and **M** will produce equilibrium.

E.g. **P** = (**– 2i + 3j**)N; **N** = (**i + 2j**)N; **M** = (**4i – j**)N; **R** = (**3i + 4j**)N.
P + N + M are in **equilibrium** since **P + N + M = 3i + 4j = - R**.

The resultant **R** is the dashed line on the diagram below.



Examine the diagram above.

The **resultant force F** = **P + N + M** = 3**i** + 4**j** = **R**

Þ -3**i** – 4**j** = -**R**.

P + N + M = (– 2**i** + 3**j**) + (**i** + 2**j**) + (4**i** – **j**) = (3**i** + 4**j**)N

and -**R** = -3**i** – 4**j**

Þ **P + N + M** + (- **R**) = **0i** + **0j** showing **equilibrium**:

(–2i + 3j) + (i + 2j) + (4i – j) + (-3i – 4j) = **0**i + **0**j

$\Rightarrow$ **no movement** from **O**

$\Rightarrow$ forces are in **equilibrium**.

Exponential function

The **general form** of the **exponential function** is **y** = $\mathbf{n^x}$, where **n** is an integer (positive) and **x** is a **variable**.

This could be described as the **index** (or power) function.

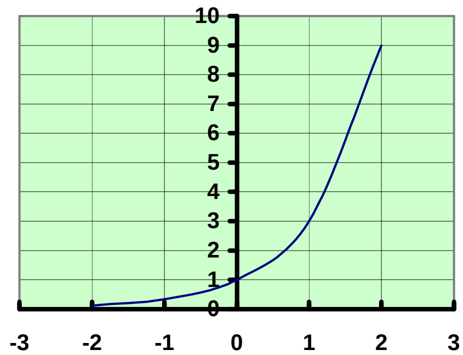
Examples of exponential functions:

(i) **y** = $\mathbf{3^x}$ gives exponential **growth**.

(ii) **y** = $\mathbf{2^{-x}}$ gives exponential **decline** (or decay).

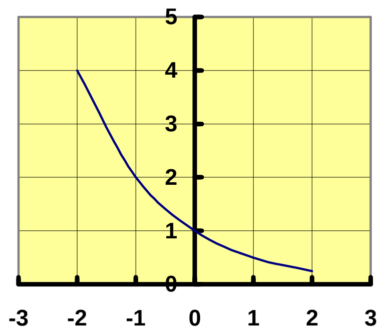
Exponential function - graph

(i) **Sketch** of $y = 3^x$



N.B. **The rate of growth may be calculated at any point on the curve by drawing the tangent to the curve at that point and determining its gradient.**

(ii) **Sketch** of $y = 2^{-x}$



N.B. **The rate of decline may be calculated at any point on the curve by drawing the tangent to the curve at that point and determining its gradient.**

Factor

A **factor** is a quantity that **divides** into another quantity, without leaving a remainder.

E.g. **2** is a **factor** of any **even** number,
or $\mathbf{a}^2$ is a **factor** of $\mathbf{a}^3$.

($2x - 1$) is a factor of $2x^2 + 3x - 2$ because $(2x - 1)(x + 2)$ is the factorized form of $2x^2 + 3x - 2$.

($2x - 3$) is a factor of $4x^2 - 9$ because $(2x - 3)(2x + 3)$ is the factorized form of $4x^2 - 9$.

Factorize

Factorizing is simply **changing** a **sum** of **terms** to a product of factors.

Factorized form can be conveniently regarded as **multiplied form**.
To **factorize** a number (or an algebraic expression), is to write it as a **product** of its **factors**.

Eg. **6**	=	**2 ´ 3**,	as a **product** of **factors**.
$\mathbf{2x + 4y}$	=	$\mathbf{2(x + 2y)}$	in **factorized** form.
$\mathbf{2x^2 + 3x - 2}$	=	$\mathbf{(2x - 1)(x + 2)}$	in **factorized** form.
$\mathbf{4x^2 - 9}$	=	$\mathbf{(2x - 3)(2x + 3)}$	in **factorized** form.

Fibonacci numbers

Fibonacci Numbers = {1, 1, 2, 3, 5, 8, 13, . . .}

Add the last two numbers to get the **next** one in the sequence.

This is known as a **recursive** sequence.

It is easy to continue:

the **next term is 21**, **then 34** and so on.

Force

(Mechanics – GCSE Additional and Advanced Subsidiary)

A **force** is a vector quantity that causes a **change** in the **state of motion** of a body.

The **unit** of force is the **newton** (**N**).

A force of 1**N** produces an acceleration of 1$\mathbf{m/s^2}$ in a body of mass 1**kg**.

The **weight** of a body is the force exerted upon it by **gravity** ($g = 9.81$ m/s^2).

Generally, the **weight** of a body of mass **m kg** is **mg N**.

E.g. A person with a ***mass*** of **60 kg** has a ***weight*** of approximately **600 N** (g is often taken as 10 m/s^2).

Forces – resolving

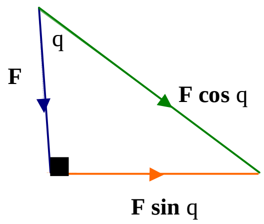
(Mechanics – GCSE Additional and Advanced Subsidiary)

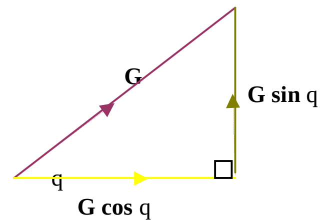
Forces can be **resolved** into **two components** at **right-angles** to each other when a **right-angled triangle** is constructed around the force, making the **force** the **hypotenuse**.

The forces are represented in **magnitude** and **direction** by the sides of the triangle in each case, using **addition of vectors**.

Look at how the forces **F** and **G** below are **each** resolved into **two components**:

(**N.B.** It is important to note the **direction of the arrow** in the force being resolved, as each diagram represents **addition of vectors**.)



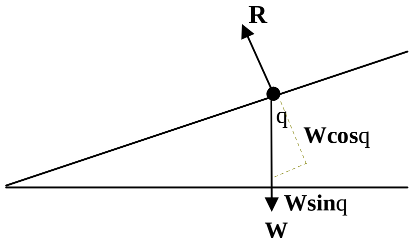


Inclined Plane

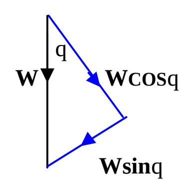
In a force diagram, a **force** can be ***replaced*** by its **components**.
(Do not show the force and its components on the same diagram.)

This is particularly helpful when dealing with an object on an inclined plane.

Force Diagram



Vector Diagram



Look at the **force diagram** and the **vector diagram** above.

The **force diagram** shows a body of weight **W** resting on a smooth plane, which is inclined at an angle q to the horizontal.
R, the **force** that the **plane exerts** on the body, acting at **right-angles** to the plane, is called the **normal reaction**.

The weight **W** can be resolved into components **perpendicular** and **parallel** to the plane, as demonstrated on the **vector diagram**. This diagram shows that the component of **W** acting *parallel* to the plane is **W sin** q and the component of **W** *perpendicular* to the plane is **W cos** q.

(These are derived from basic trigonometry on the right-angled triangle:

$$\textbf{Sin } q = \frac{\textbf{parallel}}{\textbf{W}} \quad Þ \quad \textbf{parallel} = \textbf{W sin } q$$

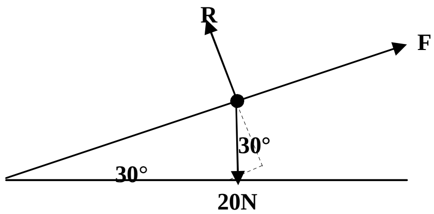
$$\textbf{Cos } q = \frac{\textbf{perpendicular}}{\textbf{W}} \Rightarrow \textbf{perpendicular} = \textbf{W cos } q.$$

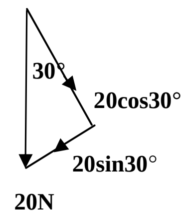
Example:

A body of weight **20N** is **at rest** on a plane, which is inclined at an angle of **30°** to the horizontal. The body is held in place by 2 forces, **F** and **R**, as shown in the **force diagram** below.

Find the 2 forces, **F** and **R**.

Force Diagram **Vector Diagram**





The force **F** is given by the **parallel** component:

⇒ F = 20 sin 30° = **10N** in the **direction of F**.

The force **R** is given by the **perpendicular** component:

⇒ R = 20 cos 30° = **17.3N** in the **direction of R**.

(NOTE: THE FORCES MUST BALANCE IN EACH DIRECTION)

Forces – resultant

(Mechanics – GCSE Additional and Advanced Subsidiary)

Two or more forces acting at a point have the same effect as a **single force**, found by **vector addition**.
This single force is called the **resultant** of the forces.

Example:

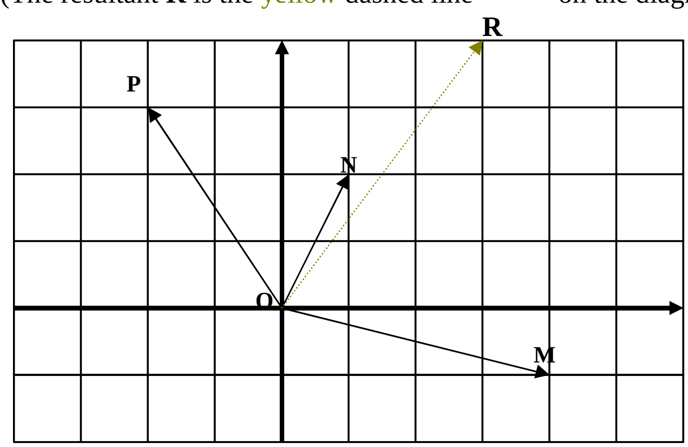
Find the magnitude and direction of the resultant force, **F** N of the forces **P** = (–2i + 3j) N, **N** = (i + 2j) N and **M** = (4i – j) N.

The **resultant force F** is given by:
(–2i + 3j) + (i + 2j) + (4i – j) = **3i + 4j**.

The **magnitude**, $|\mathbf{F}|$ = $\sqrt{3^2 + 4^2}$ = **5**.

The **direction** is **tan** $^{-1}\left|\frac{4}{3}\right|$ = **53.1°** from the **positive direction** of the **x-axis**.

(The resultant **R** is the yellow dashed line on the diagram.)



If a body is **not moving**, then the **resultant force** in any direction must be **zero**.

Hence, if **R** is the **resultant** of three forces **P**, **N** and **M**, then **–R** added to **P**, **N** and **M** will produce equilibrium.

Look again at the diagram above.

We found that **P + N + M** = $\mathbf{3i + 4j}$ = **R**
Þ $\mathbf{-3i - 4j}$ = **-R**.

We have:
$(-2i + 3j) + (i + 2j) + (4i - j) + (-3i - 4j) = \mathbf{0i} + \mathbf{0}j$
⇒ **no movement** from **O**

⇒ forces are in **equilibrium**.

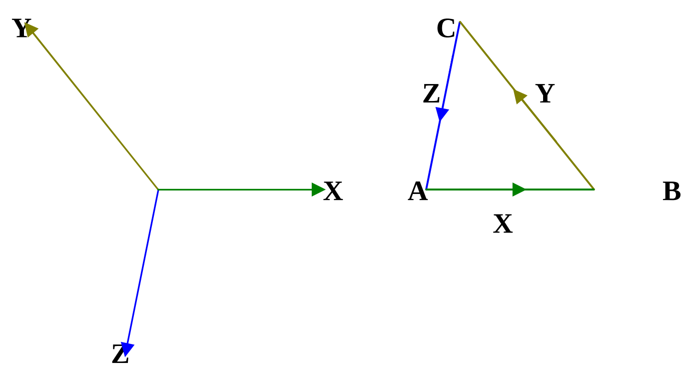
Forces – triangle of forces

If three forces acting at a point can be represented by the sides of a triangle and the **arrows** on the sides of the triangle indicating the **directions** of the forces are all in the **same sense**, then these forces are in **equilibrium**.

N.**B**. Conversely, if three forces acting at a point are in equilibrium, they can be represented by the sides of a triangle.

The triangle **ABC**, shown below, is said to be a **Triangle of Forces** for the three forces, **X**, **Y** and **Z**.

Triangle of Forces – X, Y and **Z**

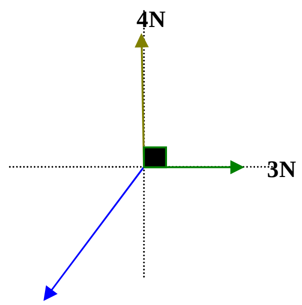


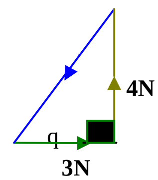
Example:

Given that the three forces shown in the diagram below are in **equilibrium**, draw a **scale** diagram and use it to find the (i) **magnitude** and (ii) **direction** of **F**.

Method:

Draw a horizontal line 3 units long, followed by a perpendicular line, 4 units long and complete the triangle. Measure the third side.





(i) The magnitude is **5**N.

(ii) The direction, q, is $\text{Tan}^{-1}\left(\frac{\mathbf{4}}{\mathbf{3}}\right)$ = **53.1°**.

(**N.B.** The sides of the triangle must be drawn **parallel** to the vectors of the forces.
Two or more forces acting at a point have the same effect as a **single force**, found by **vector addition**.
This single force is called the **resultant** of the forces.
A **body** in **motion** can **change** its **velocity** or **direction** ***only if*** a resultant force acts upon it.
If a body is **not moving**, then the **resultant force** in any direction must be **zero**.
Hence, if **R** is the **resultant** of three forces **P**, **N** and **M**, then **–R** added to **P**, **N** and **M** will produce equilibrium.

Formula

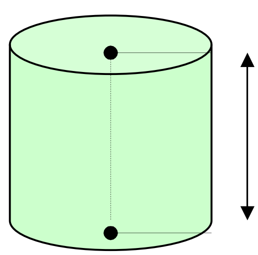
A **formula** is a **general rule**, or a fact, expressed in symbols and figures, that describes the relationship between 2 or more quantities. It is a ‘recipe’ where ‘ingredients’ must be used in a certain way to get the end result.

Examples of formulae:

(i) Area of circle = πr^2, where r is the radius.
(ii) Circumference of circle = $2\pi r$, where r is the radius.
(iii) Volume of cylinder = $\pi r^2 h$, where r is the radius and h is the perpendicular height.
(iv) Surface area of closed cylinder = $2\pi rh + 2\pi r^2$.
(Curved surface + 2 circles for top and bottom.).

Worked Example:

<u>**Cylinder**</u>:



Radius = 7cm
Height = 10cm.
Take $\boldsymbol{\pi} = 3\frac{1}{7}$.

Find (i) the **total surface area** and (ii) the **volume** of the cylinder shown.

(i) Total surface area = $2\pi rh + 2\pi r^2$.
(ii) Volume = $\pi r^2 h$.

(i) $2\pi rh + 2\pi r^2 = 2(\frac{22}{7})(7)(10) + 2(\frac{22}{7})(7^2)$.

$= \frac{2\times22\times7\times10}{7} + \frac{2\times22\times49}{7} = \frac{3080+2156}{7} = \frac{5236}{7} = 748cm^2$.

(ii) $\pi r^2 h = (\frac{22}{7})(7^2)(10) = \frac{22\times49\times10}{7} = 1540cm^3$.

Formula - **changing** the **subject** (or **transposing**)

The **steps** in **changing** the **subject** of a **formula** are:

(i) Keep the **term** (or **terms**) containing the quantity being transposed to **one side** of the **equation** and move **everything else** to the **other side**.
Remember that **brackets** can be treated as a **single term**, as the contents are together in a 'pocket'.

(ii) If there is **more than one term** containing the quantity being transposed for, **factorise** the side of the equation which contains these terms. Since it is in **each** term, it is a **common factor**, and now it appears only **once**.

(iii) When moving a quantity from one side of the equation to the other, remember that it must perform the **inverse operation** on the **other side**.

- If a quantity is doing a **multiplying** 'job' on one side, it must do a **dividing** 'job' on the other side, and vice versa.

- If a quantity is **added on** to one side, it must be **taken away** from the other side, and vice versa.

E.g. Make **x** the subject of the formula **y = 2x – 3**.

$y = 2x - 3$

$\Rightarrow \quad 2x = y + 3$

$\Rightarrow \quad x = \frac{y+3}{2} \text{ or } \frac{1}{2}(y+3)$.

We shall consider an **example** showing the way to transpose a formula for a **quantity** which **appears more than once**.

Eg. **Transpose** for ***x***: $\sqrt{\dfrac{x-y}{x+y}} \quad = \quad w.$

Square both sides to eliminate √:

$$\frac{x-y}{x+y} \quad = \quad w^2$$

Multiply both sides by (x+y) to eliminate fractions:

$$x-y \quad = \quad w^2(x+y)$$

Remove brackets: $\quad x-y \quad = \quad w^2x + w^2y$

Bring two x-terms together on one side:

$$x - w^2x \quad = \quad w^2y + y$$

Factorise – now x appears just once:

$$x(1-w^2) \quad = \quad y(w^2 + 1)$$

Divide by (1 – w ^{2}***):***

$$x \quad = \quad \frac{y(w^2+1)}{1-w^2}.$$

We now have made **x** the **subject** of the **formula**.

Fraction

A **fraction** is a bit **broken** off the **whole** of something**.**

- $\frac{1}{2}$ says **1 ÷ 2**;
  1 is the **numerator** and **2** is the **denominator**.

- $\frac{2}{4}$ is **equivalent** to $\frac{1}{2}$;
  2 is **numerator** and **4** is **denominator**.

- **Mixed numbers** are partly **whole** and partly **fraction**:

  $1\frac{1}{2}$ is **equivalent** to $\frac{3}{2}$.

- $\frac{3}{2}$ is an **improper fraction** , since its **numerator** is **greater** than its **denominator**.

- $\frac{1}{2}$ is a **proper fraction**, since its **numerator** is **less** than its **denominator**.

- **Addition** or **Subtraction**: Fractions must have **same denominator** before they can be added or subtracted:

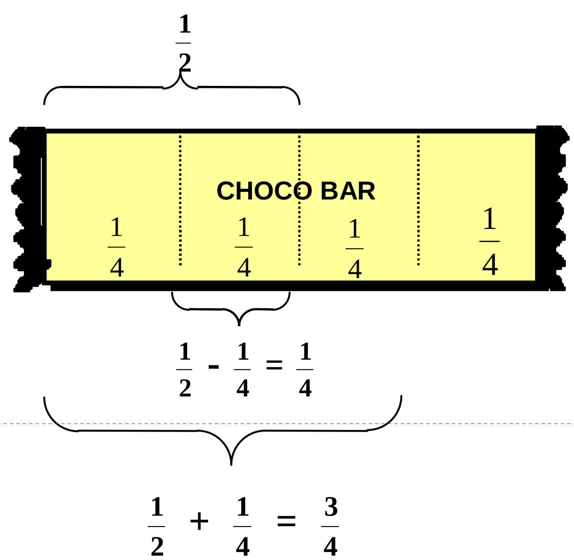
$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} \text{ and } \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

(**N.B.** $\frac{1}{2} = \frac{2}{4}$)

Look at the **Choco Bar** below.

If we wish to **add** $\frac{1}{2} + \frac{1}{4}$, we must **change the** $\frac{1}{2}$ into $\frac{2}{4}$ **before** we start.

Then : $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$



Comment [GB O'T1]:

Also, if we wish to **subtract** $\mathbf{\frac{1}{4}}$ from $\mathbf{\frac{1}{2}}$, we must **change** $\mathbf{\frac{1}{2}}$ into $\mathbf{\frac{2}{4}}$ first .

Then: $\mathbf{\frac{1}{2} - \frac{1}{4} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}}$

When **mixed numbers** are involved in **addition** or **subtraction**, the **whole number parts may be dealt with separately**, but it is often simpler to change the mixed numbers into improper fractions first and then find a **common denominator**:

$$1\frac{1}{4} + \frac{1}{2} = \frac{5}{4} + \frac{1}{2} = \frac{5}{4} + \frac{2}{4} = \frac{7}{4} = 1\frac{3}{4}.$$

$$1\frac{1}{4} - \frac{1}{2} = \frac{5}{4} - \frac{1}{2} = \frac{5}{4} - \frac{2}{4} = \frac{3}{4}.$$

- **Multiplication**: **Numerators** are **multiplied** together to give the numerator of the answer and **denominators** are **also multiplied** together to give the denominator of the answer. When mixed numbers are involved in multiplication, mixed numbers **must be changed** into **improper** fractions **before multiplying** numerators and denominators:

$$1\frac{1}{4} \times \frac{1}{2} = \frac{5}{4} \times \frac{1}{2} = \frac{5\times 1}{4\times 2} = \frac{5}{8}.$$

- **Division**: **Invert divisor** and then **multiply**.
(**Note**: Division is really upside-down multiplication.):

$$1\frac{1}{4} \div \frac{1}{2} = \frac{5}{4} \div \frac{1}{2} = \frac{5}{4} \times \frac{2}{1} = \frac{5\times 2}{4\times 1} = \frac{10}{4} = \frac{5}{2} = 2\frac{1}{2}.$$

- **Algebraic fractions – all the above rules apply**.

$$\frac{x}{2} + \frac{x}{4} = \frac{2x}{4} + \frac{x}{4} = \frac{3x}{4}.$$

$$\frac{x}{2} - \frac{x}{4} = \frac{2x}{4} - \frac{x}{4} = \frac{x}{4}.$$

$$\frac{x}{2} \times \frac{x}{4} = \frac{x^2}{8}.$$

$$\frac{x}{2} \div \frac{x}{4} = \frac{x}{2} \times \frac{4}{x} = \frac{4x}{2x} = 2.$$

$$\frac{2}{5} \text{ x } (\frac{3}{2}\text{x} + \frac{1}{4}\text{x}) = \frac{2x}{5} \times \frac{3x}{2} + \frac{2x}{5} \times \frac{x}{4}$$

$$= \frac{6x^2}{10} + \frac{2x^2}{20} = \frac{12x^2}{20} + \frac{2x^2}{20} = \frac{14x^2}{20} = \frac{7x^2}{10} \text{ (or } \frac{7}{10}x^2\text{)}.$$

Frequency In statistics, the frequency of a score is the number of times, i.e. how ***frequently***, that score occurs.

E.g. In the set of scores 2, 9, 3, 2, 3, 7, 9, 3, 2 and 4, the **frequencies** for each score are as follows:

Score	**2**	**3**	**4**	**7**	**9**
Frequency	**3**	**3**	**1**	**1**	**2**

Frequency Density

<u>**Frequency Density**</u> = $\dfrac{\textbf{Frequency}}{\textbf{ClassWidth}}$.

E.g. A survey was conducted on **60** people to find out how many newspapers each person bought in a week.
The results are shown in the frequency table below.
Work out the **frequency density** for each score or group of scores shown in the table.

Score	0, 1 or 2	3	4	5	6	7	8 or 9
Frequency	9	8	12	6	13	8	4
Frequency Density	$\frac{9}{3}=3$	$\frac{8}{1}=8$	$\frac{12}{1}=12$	$\frac{6}{1}=6$	$\frac{13}{1}=13$	$\frac{8}{1}=8$	$\frac{4}{2}=2$

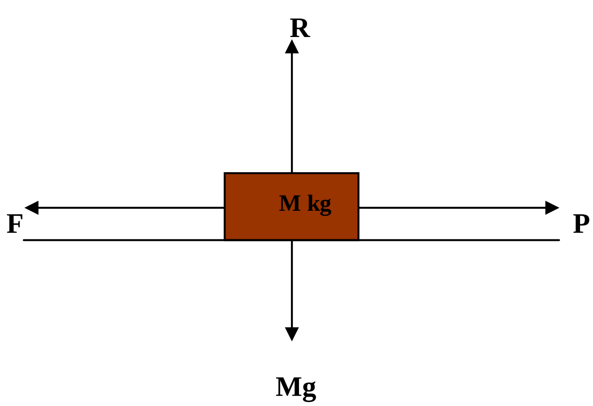
(N.B. 9 divided by 3) **(N.B. 4 divided by 2)**

Friction

(Mechanics – GCSE Additional and Advanced Subsidiary)

If a body of mass **M** kg rests on a horizontal surface, and a horizontal force of **P** N is applied to the body, **equal** and **opposite** forces act on the body and on the plane at **right-angles** to the surfaces in contact. *(Newton's Third Law).*

$\therefore$ **R = Mg**



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The force, **F**, *opposing* the motion of the body, is called the **frictional force**.

The frictional force acts in the **opposite** direction to the motion and is **parallel** to the surfaces that are in contact.

With perfectly **smooth** surfaces, there is **no** frictional force, i.e. **F = 0**, so the body would move no matter how small is the applied force **P**.

With **rough** surfaces, the body will move *only if* **P**, the applied force, is **greater** than **F**, the frictional force.

The magnitude of the frictional force depends on the roughness of the surfaces and the force **P**, which is trying to move the body.

For **small** values of **P**, there is no movement of the body and **F = P**.

As the applied force **P** increases, the frictional force **F** increases until **F** reaches a **maximum value F(max)**, beyond which it cannot increase.

At this point, the body is in a state of **limiting equilibrium**, and is *on the point* of moving.

The magnitude of **F(max)** is a **fraction** of the normal reaction **R**.

This **fraction** is called the **coefficient** of **friction** and is denoted by m
We have: **F(max)** = m**R** for the two surfaces in contact.

Graph of f(x) - simple transformations

The graph of **y = f(x)** can be easily **transformed** as follows:

(i) **f(x) + a**, where **a** is a constant, by moving **f(x)** through **a** units **vertically**.

Eg. **f(x) + 2** moves **f(x)** *upwards* through **2** units.
f(x) - 2 moves **f(x)** *downwards* through **2** units.

(ii) **f(x + a)**, where **a** is constant, by moving **f(x)** through **- a** units **horizontally**.

Eg. **f(x + 2)** moves **f(x)** *leftwards* through **2** units.
f(x – 2) moves **f(x)** *rightwards* through **2** units.

(iii) **af(x)**, where **a** is a constant, by **stretching f(x)** by a scale factor of **a** units **vertically**.

Eg. **2f(x)** multiplies the **y-value** at each **f(x)** point by **2**.

(iv) **f(ax)**, where **a** is a constant, by multiplying the **width** of **f(x)** by a scale factor of $\frac{1}{a}$.

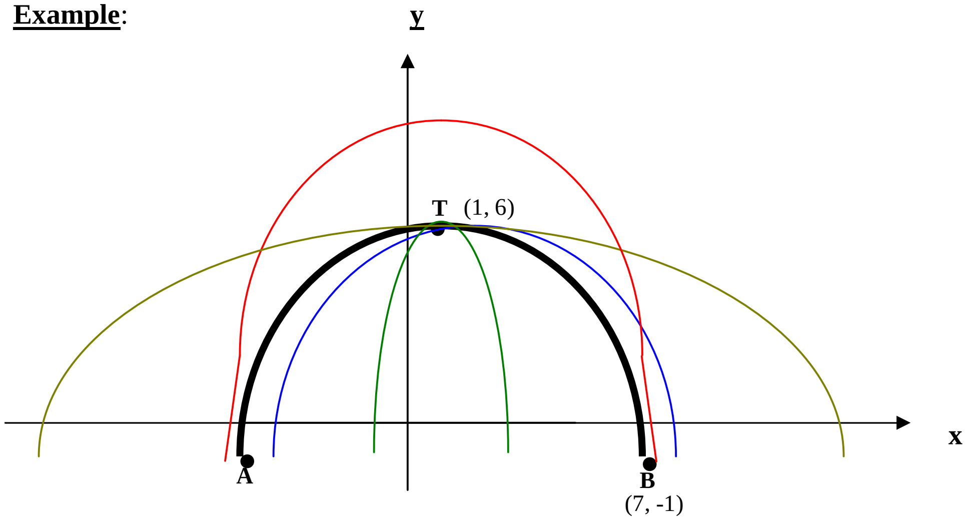
Eg. **f(2x)** multiplies the **x - value** of each point on **f(x)** by

$\frac{1}{2}$, i.e. the **width** of **f(x)** is **halved**.

f($\frac{1}{2}$x) multiplies the **x - value** of each point on **f(x)** by

2, i.e. the **width** of **f(x)** is **doubled**.

Example:



On the diagram above, the quadratic graph **y = f(x)** is shown in **black**. **f(x)** has its maximum turning point at **T(1, 6)** and the points **A** and **B** are on the curve of f(x). The point **B** has coordinates (**7**, **-1**).
On the same diagram, using the same scales, sketch the graphs of each of the following, stating the coordinates of **T**, **A** and **B** following each transformation:

		T	**A**	**B**
(i)	f(x) + 3. Shown in red. (f(x) moved 3 units upwards).	(1, 9)	(-5, 2)	(7, 2)
(ii)	f(x - 1). Shown in blue. (f(x) moved 1 unit rightwards).	(2, 6)	(-4, -1)	(8, -1)
(iii)	f(3x). Shown in green. (f(x) is divided by 3 widthwise).	(1, 6)	(-1, -1)	(3, -1)
(iv)	f(½x). Shown in dark yellow. (f(x) is doubled in width).	(1, 6)	(-11, -1)	(13, -1)

Highest Common Factor (H.C.F.)

The **highest common factor (H.C.F.)** is the **highest** quantity that will **divide exactly** into **each** of a group of quantities.

E.g.(i) **6** is the **H.C.F.** of **12**, **18** and **24**.

E.g.(ii) $\mathbf{ac^2}$ is the **H.C.F.** of $\mathbf{a^2b^2c^2}$, $\mathbf{ac^3}$ and $\mathbf{a^3b^4c^4}$.

E.g.(iii) To **factorize** $\mathbf{3x^2y^4 - 6x^3y^3}$ the H.C.F. $\mathbf{3x^2y^3}$ is required, giving $\mathbf{3x^2y^3(y - 2x)}$ as the factorized form of $\mathbf{3x^2y^4 - 6x^3y^3}$.

Histogram A **histogram** is the display of data in the form of a block graph, where the **area** of each **rectangle** is **proportional** to the **frequency**.

E.g. A survey was conducted on **60** people to find out how many newspapers each person bought in a week.

The results are shown in the **frequency table** below.

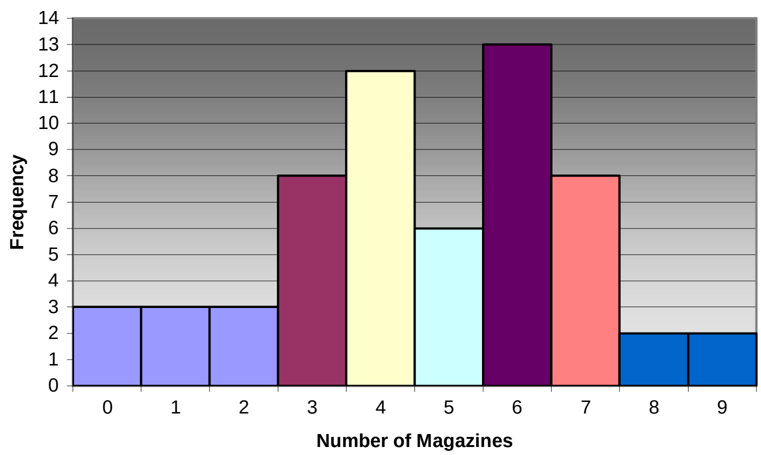
Compile a **histogram** to represent the data.

Score	0, 1 or 2	3	4	5	6	7	8 or 9
Frequency	9	8	12	6	13	8	4

(N.B. 9 must be divided by 3) **(N.B. 4 must be divided by 2)**

N.B. **Frequency Density** = $\dfrac{\textbf{Frequency}}{\textbf{ClassWidth}}$

HISTOGRAM



Histogram for a Grouped Distribution

A **histogram** for a **grouped distribution** must have the **mid-points** of the **class intervals** at the **centre** of each **rectangle**, representing the **frequency**. It must be emphasised that the **width** of each **rectangle** is the **difference** between **upper** and **lower class boundaries**, **NOT** the difference between **upper** and **lower class limits**.

The **table** below gives a **grouped frequency distribution** for the **heights** (to the nearest cm) of **100** fifth-year pupils in a secondary school:

Class Interval	Height (cm)	Frequency	Cumulative Frequency	Mid-point of ClassInterval
1st	**145-149**	**5**	**5**	**147**
2nd	**150-154**	**8**	**13**	**152**
3rd	**155-159**	**15**	**28**	**157**
4th	**160-164**	**35**	**63**	**162**
5th	**165-169**	**20**	**83**	**167**
6th	**170-174**	**10**	**93**	**172**
7th	**175-179**	**7**	**100**	**177**
	<u>TOTAL</u>	**<u>100</u>**		

Since these heights have been rounded off to the **nearest cm**, in theory, each **class interval** contains heights of up to **0.5cm above** and **below** the **class limits**:

E.g. **Class 1**, **145-149** contains heights between **144.5cm** and **149.5cm**.

These figures give the **upper** and **lower class boundaries**.

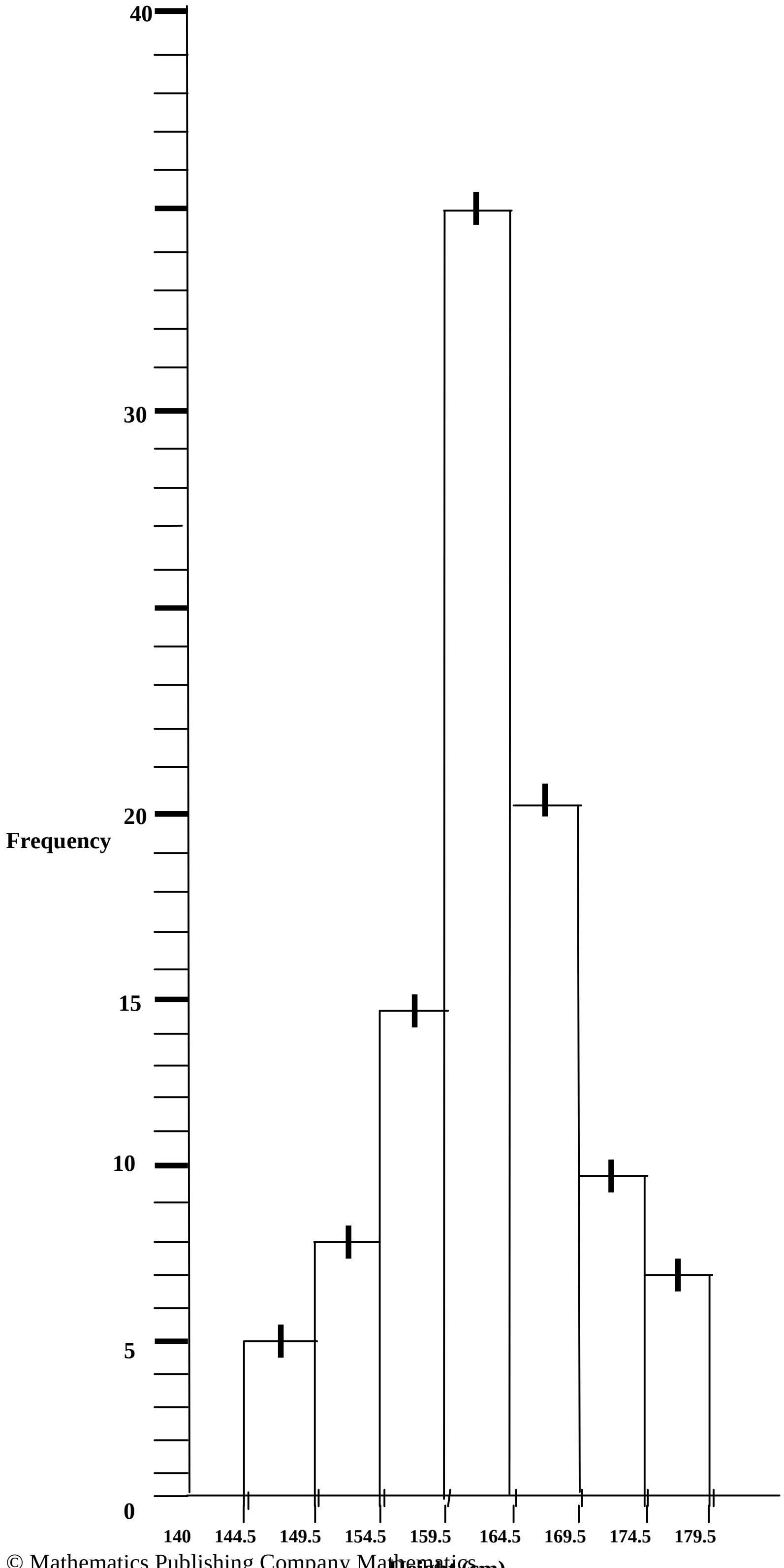
It is important to note that each **class interval** has **width 5cm**, <u>not</u> **4cm**, i.e. the **difference** between the **upper** and **lower class boundaries;** **149.5 - 144.5 = 5**, **154.5 – 149.5 = 5**, …

This is particularly important when finding the **mid-point** of a **class interval**, which is required to find the **mean** of a distribution, or to draw a **histogram**, in which the **mid-points** of the **class intervals** must be at the **centres** of the **rectangles**.

To find the **mid-point** of a **class interval**, simply **add** the **upper** and **lower class boundaries** and **divide** by **2**.

Please examine the **histogram** below for the heights of the **100** pupils detailed in **Table 3**. Remember that the **width** of each **rectangle** is **5cm**, not **4cm** (i.e. 149.5-144.5, 154.5-149.5,…).

Heights (to nearest cm) of 100 Fifth-Year Pupils



Website: www.mathslearning.com ; Email: mathsbooks@hotmail.co.uk

Impulse and Momentum (Mechanics – GCSE Additional and Advanced Subsidiary)

The **momentum** of a body is: **Mass ´ Velocity**.

If the units of mass and velocity are **kg** and **m/s** respectively,
then the **units** of **momentum** are **newton-seconds** (**N s**).

Since the momentum of a body depends upon its velocity,

momentum is a vector quantity.

The **impulse** of a constant force **F** is:

F ´ t, where **t** is the time for which the force is acting.

F = ma gives: $Ft = ma \times t$

v = u + at gives: $Ft = m(v - u)$

$\therefore$ $Ft = mv - mu$

Therefore: **Impulse = Change in momentum**.

Example 1:

Find the magnitude of the momentum for each of the following:

(i) A lorry of mass10 tonnes moving with a speed of 25m/s.
(ii) A ball of mass 250g moving with a speed of 12m/s.
(iii) A girl of mass 55kg moving with a speed of 3m/s.

Method:

(i)	Momentum	=	mass $\times$ velocity
		=	$(10 \times 1000) \times 25$
		=	250000 N s.
(ii)	Momentum	=	mass $\times$ velocity
		=	$(250 \div 1000) \times 12$
		=	3 N s.
(iii)	Momentum	=	mass $\times$ velocity
		=	55×3
		=	165 N s.

If the **velocity** of a body changes from **u** to **v**, then its momentum also changes.

The initial momentum is **mu** and the final momentum is **mv**.

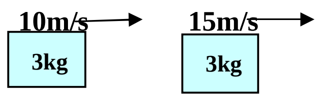
Example 2:

Find the change in momentum of a body of mass 3kg when its speed changes:

(i) from 10m/s to 15m/s in the same direction.
(ii) from 8m/s to 5m/s in opposite directions.

Method: Draw 2 diagrams: **Initial** **Final**

Taking velocities to right as +:



(i)
Initial Momentum = $3 \times 10 = 30$ N s

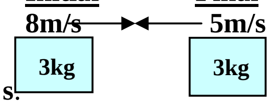
Final Momentum = $3 \times 15 = 45$ N s

$\therefore$ Change in momentum = **15 N s**.

Initial **Final**

Taking velocities to right as +:

(ii)



Initial Momentum = 3×8 = **24 N s**.

Final Momentum = $3 \times (-5)$ = -15 N s

$\therefore$ Change in momentum = **39 N s**.

Impulse = Change in momentum.

When **two particles collide**, each receives an impulse from the other **equal** in **magnitude** but **opposite** in **sign**. The sum of the momenta before impact is equal to the sum of the momenta after impact. (*The principle of Conservation of Linear Momentum.*)

The **total change** in momentum is, therefore, **zero**.

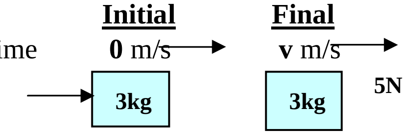
Example 1:

A body of mass **3kg** is initially at rest on a smooth horizontal surface. A horizontal force of **5N** acts on the body for **8** seconds.

Find: (i) the magnitude of the impulse given to the body.
(ii) the magnitude of the final momentum of the body.
(iii) the final speed of the body.

Method:

(i) Draw 2 diagrams: **Initial** **Final**
Impulse = Force $\times$ Time
$= 5 \times 8 =$ **40 N s**.



(ii) Initial Momentum, mu = 0

Impulse = mv – 0

$\therefore$ 40 = mv, i.e. final momentum.

(iii) mv = 40
$\therefore$ 3v = 40
$\therefore$ v = $\mathbf{13\frac{1}{3}}$, i.e. final speed.

Index

A quantity raised to a **power** (or **index**) is **multiplied** by **itself** the **number** of **times** that the **power** states.

Thus $\mathbf{x^2 = x \times x}$, $\mathbf{x^3 = x \times x \times x}$ and $\mathbf{x^5 = x \times x \times x \times x \times x}$.

In these examples, the **powers** (or **indices**) are **2**, **3** and **5**.

Index Laws

Since a quantity raised to a **power** (or **index**) is **multiplied** by **itself** the **number** of **times** that the **power** states, clearly then $\mathbf{x^2 = x \times x}$, $\mathbf{x^3 = x \times x \times x}$, $\mathbf{x^5 = x \times x \times x \times x \times x}$ and so on.
The **powers** (or **indices**) are **2**, **3** and **5** and we can see that writing down the quantity **x** the number of times that the power states and then multiplying all the **x**s together, is all that is required.
It follows that $\mathbf{2^4 = 2 \times 2 \times 2 \times 2 = 16}$ and $\mathbf{3^3 = 3 \times 3 \times 3 = 27}$.

We shall now look at a set of **laws** which can be employed when working with **indices**.

(i) **The Multiplication Law**
\- **add the indices**

E.g. $\mathbf{x^2 \times x^3 = x^{2+3} = x^5}$.

Writing $\mathbf{x^2 = x \times x}$ and $\mathbf{x^3 = x \times x \times x}$, we have:
$\mathbf{x^2 \times x^3 = x \times x \times x \times x \times x = x^5}$.

(ii) **The Division Law**
\- **subtract** the **denominator index** from the **numerator index**

E.g. $\mathbf{x^5 \div x^2 = x^{5-2} = x^3}$.

Writing $\mathbf{x^5 = x \times x \times x \times x \times x}$ and $\mathbf{x^2 = x \times x}$, we have:

$$\mathbf{\frac{x \times x \times x \times x \times x}{x \times x} = x^3}.$$

(iii) **The Powers Law**
\- **multiply the indices**

E.g. $\mathbf{(x^2)^3 = x^{2 \times 3} = x^6}$.

Write $\mathbf{(x^2)^3 = x^2 \times x^2 \times x^2 = x \times x \times x \times x \times x \times x = x^6}$.

(iv) **Negative Index**

A negative index indicates the **reciprocal** of the quantity, i.e. the '**upside-down**' version.

E.g. **2** can be written as $\mathbf{\frac{2}{1}}$ and the reciprocal of **2** is $\mathbf{\frac{1}{2}}$.

Also $\mathbf{x}^{-2}$ can be written as $\frac{\mathbf{x}^{-2}}{\mathbf{1}}$ and then inverting gives $\frac{\mathbf{1}}{\mathbf{x}^2}$.

Note that $\frac{\mathbf{1}}{\mathbf{x}^{-2}} = \frac{\mathbf{x}^2}{\mathbf{1}} = \mathbf{x}^2$.

The general rule is that the **sign** of the **index changes** if the **x-term** moves up or down past the bar separating the numerator from the denominator.

FURTHER EXAMPLES

(a) $\frac{\mathbf{1}}{\mathbf{x}^{-4}} = \frac{\mathbf{x}^4}{\mathbf{1}} = \mathbf{x}^4$.

(b) $\mathbf{2x}^{-3} = \mathbf{2} \times \mathbf{x}^{-3} = \mathbf{2} \times \frac{\mathbf{1}}{\mathbf{x}^3} = \frac{\mathbf{2}}{\mathbf{x}^3}$.

(c) $(\mathbf{2x})^{-3} = \frac{\mathbf{1}}{(\mathbf{2x})^3} = \frac{\mathbf{1}}{\mathbf{8x}^3}$.

Note the difference between (b) and (c) above.
In (c) the 2 is unaffected by the power, whereas in (d), everything in the brackets is raised to the power of –3.

Sometimes a negative index results after the application of one of the other laws of indices:

For example, $x^2 \times x^{-5} = x^{2+(-5)} = x^{-3} = \frac{1}{x^3}$,

$x^2 \div x^4 = x^{-2} = \frac{1}{x^2}$ and $(x^{-2})^3 = x^{-6} = \frac{1}{x^6}$.

(v) **Zero Index**

Any **non-zero** number raised to the **power 0** equals **1**.

Take $\mathbf{x}^3 \div \mathbf{x}^3 = \frac{\mathbf{x}^3}{\mathbf{x}^3} = \mathbf{x}^{3-3} = \mathbf{x}^0 = \mathbf{1}$.

Generally, then, for any quantity **x**:

$$\frac{\mathbf{x}^n}{\mathbf{x}^n} = \mathbf{x}^{n-n} = \mathbf{x}^0 = \mathbf{1}.$$

It follows that $\mathbf{10}^0 = \mathbf{1}$, $\mathbf{1000000}^0 = \mathbf{1}$, $(\frac{\mathbf{1}}{\mathbf{2}})^0 = \mathbf{1}$, to mention only three examples.

(vi) **Fractional Indices**

Fractional indices indicate **roots**, and the **denominator** of the fraction tells us which root to take.

E.g. $\mathbf{x^{\frac{1}{2}} = \sqrt{x}}$, $\mathbf{x^{\frac{1}{3}} = \sqrt[3]{x}}$, $\mathbf{x^{\frac{1}{4}} = \sqrt[4]{x}}$, etc.

The reasons for this can be explained if we consider:

$$\mathbf{x = x^1 = x^{\frac{1}{2}} \times x^{\frac{1}{2}} \quad \Rightarrow \quad x^{\frac{1}{2}} = \sqrt{x}},$$

$$\mathbf{x^1 = x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} \quad \Rightarrow \quad x^{\frac{1}{3}} = \sqrt[3]{x}},$$

$$\mathbf{x^1 = x^{\frac{1}{4}} \times x^{\frac{1}{4}} \times x^{\frac{1}{4}} \times x^{\frac{1}{4}} \quad \Rightarrow \quad x^{\frac{1}{4}} = \sqrt[4]{x}} \ldots$$

Examples:

$\sqrt{\mathbf{25}} = 25^{\frac{1}{2}} = 5$, since $5^2 = 25$,

$\sqrt[3]{\mathbf{125}} = 125^{\frac{1}{3}} = 5$, since $5^3 = 125$,

$\sqrt[4]{\mathbf{16}} = 16^{\frac{1}{4}} = 2$, since $2^4 = 16$, and so on.

$27^{\frac{2}{3}} = (27^{\frac{1}{3}})^2 = (\sqrt[3]{\mathbf{27}})^2 = 3^2 = 9.$

$\sqrt{\mathbf{25^3}} = (25^3)^{\frac{1}{2}} = 25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 = 5^3 = 125.$

$$32^{-\frac{4}{5}} = \frac{1}{32^{\frac{4}{5}}} = \frac{1}{(32^{\frac{1}{5}})^4} = \frac{1}{2^4} = \frac{1}{16}.$$

Integration (GCSE Additional Pure and Advanced Subsidiary Pure)

Integrating a function is equivalent to finding the **area under** the **curve**.

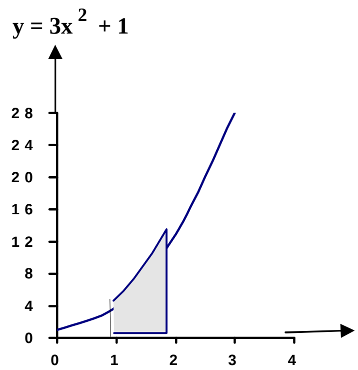
The **technique** for **integrating** a function is as follows:
1. **Add 1** to **index**.
2. **Divide** by **new index**.
3. Allow for the presence of a **constant** in the original function by adding '**+ c**'; this is known as the '**constant** of **integration'**.
4. **Coefficients** are **not affected** by integration.

The integral $\int \mathbf{y\,dx}$ gives the **general area** between the **curve** of **y** and the **x-axis** – this is an **indefinite** integral.
When **limits** are known, an **actual area** may be calculated – this is a **definite** integral.
In a definite integral, there is **no need** for **c**, the constant of integration.

E.g. Find $\int_1^2 \mathbf{(3x^2 + 1)dx}$.

(This means: find the **area** bounded by the curve, $\mathbf{y = 3x^2 + 1}$, the **x-axis** and the lines $\mathbf{x = 1}$ and $\mathbf{x = 2}$.)

The diagram looks like this, with the required area shaded:



$\int_1^2 (3x^2 + 1)dx = \left[x^3 + x\right]_1^2$. Notice the notation - [ ] is used once the integration has been done and the limits for x are moved to the right side.

Now, **evaluate** this integral using the limits, **1** and **2**:

$\left[2^3 + 2\right] - \left[1^3 + 1\right] \quad = \quad$ **10 - 2 = 8** sq. units for shaded region.

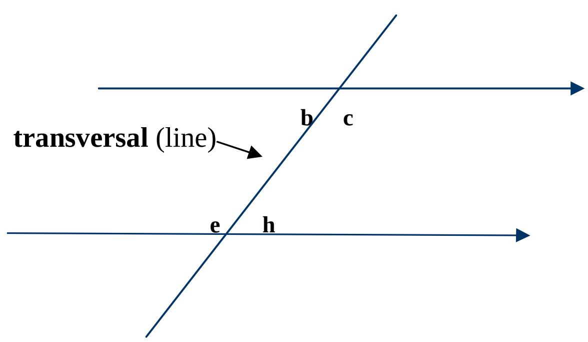
Interior Angle

Interior angles are the angles on the **inside** between **each** of two parallel lines and a **transversal** drawn through these parallel lines.

Interior angles are **supplementary** (i.e. **add up** to **180°**)

Interior angles in the diagram below:

b + e = 180° **c + h = 180°**



Interquartile range

See **Cumulative Frequency diagram** (Ogive) below.

If we divide the **total frequency** into **quarters**, the **score** corresponding to the **lower quarter** is the **lower quartile** and the **score** corresponding to the **upper quarter** is the **upper quartile**.
The **interquartile range** is the **difference** between the **score readings** provided by the **upper** and **lower quartiles**.

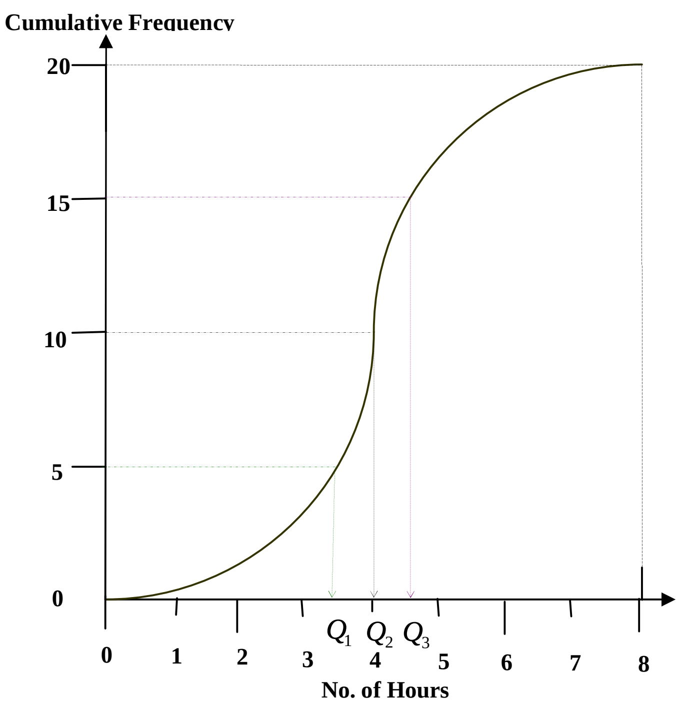
E.g. In a survey, **20** children were asked how many hours they spend on sports in each week.
The results of the survey were as follows:

4	**5**	**1**	**4**	**5**
3	**4**	**2**	**6**	**4**
4	**4**	**5**	**5**	**7**
5	**4**	**5**	**8**	**5**

Cumulative Frequency Table from given data

Number of Hours (less than or equal to)	**Cumulative Frequency**
0	**0**
1	**1**
2	**2**
3	**3**
4	**10**
5	**17**
6	**18**
7	**19**
8	**20**

The **cumulative frequency diagram** (ogive) compiled from the above data looks like this:



The **interquartile range** is the **difference** between the **score readings** provided by the **upper** and **lower quartiles**.

Lower Quartile, Q_1 = **3.4**; **Upper Quartile**, Q_3 = **4.6**.
Interquartile range = 4.6 – 3.4 = 1.2 hours.

Intersecting graphs – solution of simultaneous equations

Equations may be solved **graphically** by using **intersecting graphs**. If two functions are plotted on the **same axes**, using the **same scales**, the functions are **equal** to each other at the **point**(**s**) of **intersection** between their **graphs**.

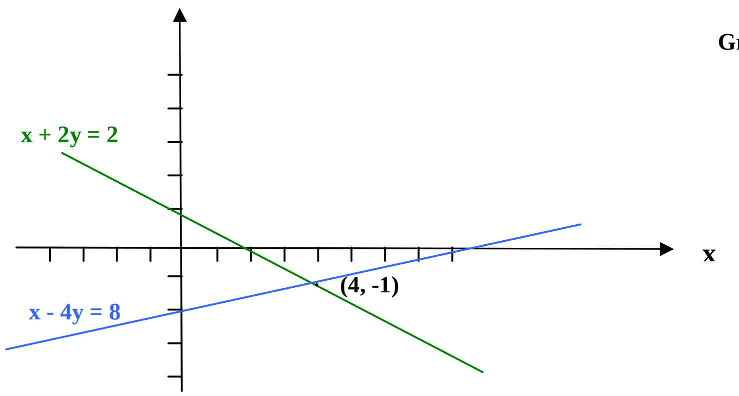
The examples below demonstrate the method.

Linear Equations

Use **graphical methods** to solve the following simultaneous equations:

(a) $\mathbf{x + 2y = 2}$
$\mathbf{x - 4y = 8}$.

Graph (i) (a)



The point of intersection is (**4**, **-1**), giving the solution.

(b) Draw the graph of $\mathbf{y = x^2}$ and use your graph to solve
$\mathbf{x^2 - x - 2 = 0}$,
by *drawing an appropriate straight line* on your graph.

$\Rightarrow \quad \mathbf{x^2 = x + 2}$.
$\Rightarrow \quad \mathbf{x^2 = x + 2}$.
$\Rightarrow \quad \mathbf{y = x + 2}$ is the **required line**.

The points of intersection between the curve $\mathbf{y = x^2}$ and the line $\mathbf{y = x + 2}$ are (**2**, **4**) and (**-1**, **1**), giving $\mathbf{x = 2}$ or $\mathbf{x = -1}$ as the solution.

Irrational number

An irrational number is any number that **cannot** be written as a **fraction**, $\frac{numerator}{denominator}$, for example π, $\sqrt{2}$, $\sqrt{3}$, … are irrational.

Since irrational numbers can never be written **accurately** in decimal form, they can never be expressed accurately in fraction form either.

Iteration

E.g. (i) Show that $\mathbf{3x^2 + x - 2 = 0}$ can be written as $\mathbf{x = \pm\sqrt{\frac{2 - x}{3}}}$.

(ii) Taking the **positive** sign, write down the corresponding **iteration formula** (giving $\mathbf{x_{n+1}}$ in terms of $\mathbf{x_n}$).

(iii) Use this iteration formula to find, correct to **2 decimal places**, the **positive root** of the equation $\mathbf{3x^2 + x - 2 = 0}$.

$$3x^2 + x - 2 = 0$$
$$\Rightarrow \quad 3x^2 = 2 - x$$
$$\Rightarrow \quad x^2 = \frac{2 - x}{3}$$
$$\therefore \quad x = \pm\sqrt{\frac{2 - x}{3}}.$$

Q.E.D. (Latin: *Quod erat demonstrandum* means '*what was to be shown'*).

(ii) $\mathbf{x_{n+1} = \sqrt{\frac{2 - x_n}{3}}}$ is the iteration formula.

(iii)

x_1	=	**0.5**	or	**0.9**
x_2	=	**0.7071**		**0.6055**
x_3	=	**0.6565**		**0.6818**
x_4	=	**0.6692**		**0.6629**
x_5	=	**0.6660**		**0.6676**

$\therefore$ **x** = **0.67**, correct to **2 decimal places** gives the **positive root** of the equation: $\mathbf{3x^2 + x - 2 = 0}$.

Leibniz, Baron von

(See **Calculus**, **Differentiation** and **Integration**.)

Baron von Leibniz (1646 – 1716), a German scholar, mathematician and philosopher, shares with Sir Isaac Newton the distinction of developing the theory of the **differential** and **integral calculus**; his **notation** was adopted, in favour of Newton's.

$\mathbf{\frac{dy}{dx}}$ and $\int$ **dx** are **Leibnizian** notation for **derivative** and **integral** respectively.

Like Terms

Since all terms containing **x,** for example**,** in any algebraic expression are **the same**, they can be **collected together** to form a **single term** in **x**, by **adding, subtracting, multiplying** and **dividing** as required. The **terms** that are **alike** are called **like terms** and adding and subtracting them to form a **single term** in **x** is **collating like terms**. To **express** an **algebraic** expression in its **simplest** form, **like terms must be collated**.

Eg.(i) **Simplify** **2x - 3x + 5x - 4x + x + 1 –5 + 2**.

Collating like terms, we have:
+2x –3x + 5x –4x +1x = + 1x = **x** and **+ 1 – 5 + 2 =** **-2**,
giving **x – 2** as the **simplest form**.

Eg.(ii) **Simplify** **2x + 1 + x – 3 - 4x + 5 – x – 10**.

Collating like terms, we have:
+**2x +1x –4x –1x = -2x** and **+1 –3 +5 = +3**, giving **-2x +3** as the **simplest form**.

Eg.(iii) **Simplify** **-xy +x –2xy +3 –4x –7 +z –5z +p**.

Collating like terms, we have:
-1xy –2xy = -3xy,
+1x – 4x = -3x,
+ 3 - 7= - 4
and **+1p = + p**,
giving **- 3xy – 3x – 4 + p** as the **simplest form**.

Eg. (iv) Write the following in its **simplest form**, by **firstly removing** the **brackets**, and then **collating like terms**:
2*x* –7 – (3*x* + 5) + 3*y* –2 +5(2*y* – 2) +(*y* – 4) +10.
N.B. -(3*x* + 5) is **–1**(3*x* + 5) and +(*y* – 4) is +**1**(*y* – 4).

Removing brackets gives:
2x – 7 – 3x – 5 +3y – 2 + 10y – 10 + y – 4 + 10.
Collating x terms, we have:
2x – 3x = -1x = -x.
Collating y terms we have:
+3y + 10y + 1y = +14y.
Collating numbers, we have:
-7 –5 –2 –10 –4 +10 = -18.

The **simplest form** of the **whole expression** is, therefore:
-x + 14y – 18.

Linear programme

Linear Programming is a useful application of systems of linear inequalities of the form **y** < **mx + c**, **y** > **mx + c** or **y = mx + c**.
Similar systems are used in business management, where there is a need to determine maximum profits or minimum costs, in compliance with certain conditions laid down.
A **linear inequality** is set up to fit **each condition**, and the **whole system** of inequalities, when plotted on a graph, encloses a **polygon**, which is the **feasible region** or **solution set**.
This means that every point **(x**, **y)** in this region is feasible under the conditions laid down, but the **maximum** or **minimum** values will occur at, or close to, the **vertices** of the polygon, or at all points along one of its edges.

SAMPLE QUESTION

A lady is following a special diet which specifies a minimum daily consumption of 84 units of protein, and a minimum daily consumption of 36 units of carbohydrate. To meet these requirements she must eat portions of yogurt and portions of meat. The protein and carbohydrate content of a portion of each food and the corresponding cost are set out in the table below:

Portion	Units of Protein	Units of Carbohydrate	Cost
Yogurt	5	4	£0.50
Meat	12	3	£1.25

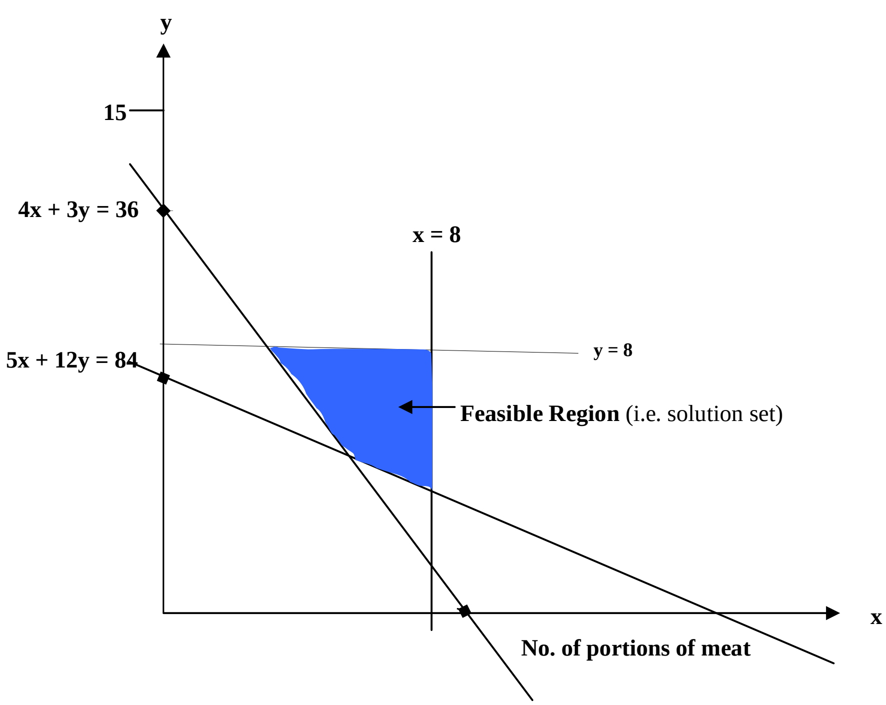
The lady takes **x** portions of **yogurt** and **y** portions of **meat** per day and the number of portions of each must not exceed **8** per day.

(i) Write down four inequalities involving **x** and/or **y**, other than **x** ³ **0** and **y** ³ **0**.

The four inequalities are listed below.

Condition I: Protein: 5x + 12y ³ 84
Condition II: Carbohydrate: 4x + 3y ³ 36
Condition III:No. of Portions: x £ 8
y £ 8.

(ii) Using a scale of 1cm to represent one unit on each axis, illustrate these four inequalities on a single diagram on graph paper.
See the diagram below.



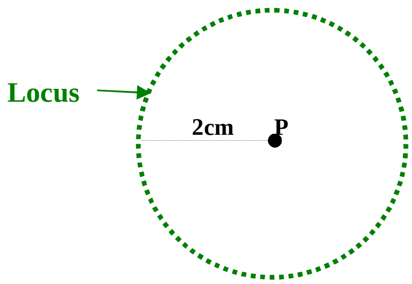
Locus (plural ***Loci***)

A **locus** (Latin: *locus* means *path* or *place*) is a path traced out by a point moving in accordance with a certain law.

Examples:

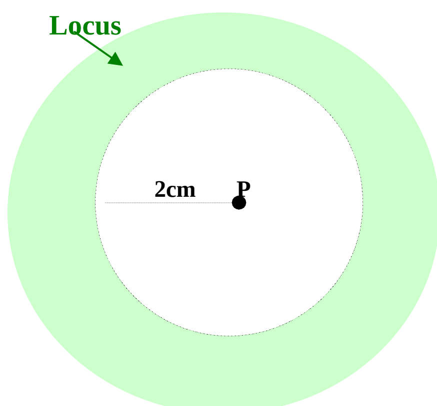
(i) The locus of a point that moves so that it is always **2cm** away from a **fixed point P** is a **circle** of **radius 2cm**, centre **P**.

Hint: It is always helpful to mark a few points according to the given law in order to gain some idea of how the locus looks.

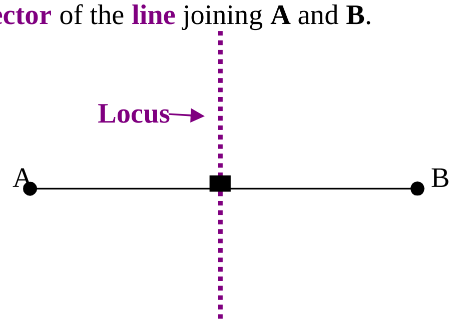


(ii) The locus of a point that moves so that it is always **more than 2cm** away from a **fixed point P - all points** in the **region outside** a **circle** of **radius 2cm**, centre **P**.

Hint: It is always helpful to mark a few points according to the given law in order to gain some idea of how the locus looks.



(iii) The locus of a point moving so that it is always a fixed distance away from two fixed points **A** and **B** is the **perpendicular bisector** of the **line** joining **A** and **B**.



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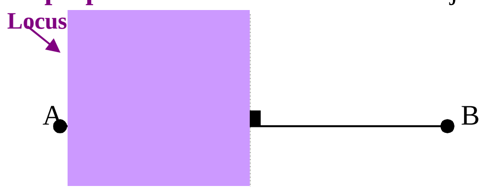
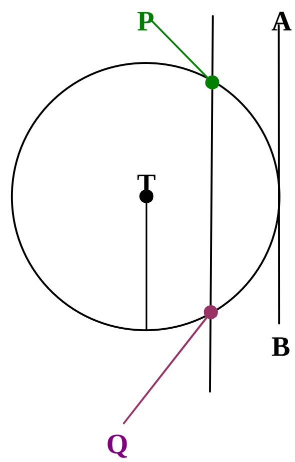
(iv) The locus of a point moving so that it is always **closer** to **A** than to **B** in the diagram above **- all points** in the **region** to the **left** of the **perpendicular bisector** of the **line** joining **A** and **B**.



Intersecting Loci

When two (or more) pieces of information are given about the position of a point, each condition is dealt with *separately*, and the **intersection** of the **loci** gives the required position of the point(s):

E.g. Two points, **P** and **Q** are **2cm** from **T** and **1cm** from the line **AB**. Mark the positions of **P** and **Q** on the diagram below:



Lower bound

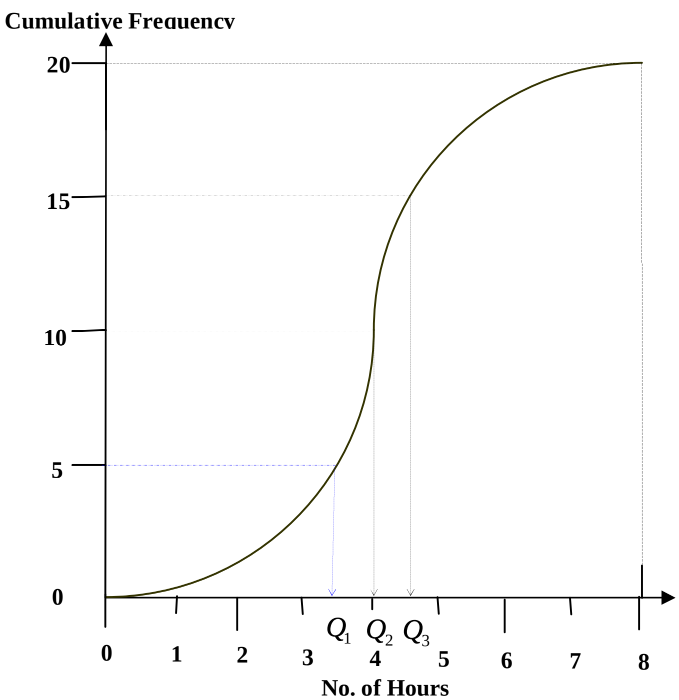
A number rounded off to so many decimal places or significant figures is **not** an **accurate** representation of the number – there is a **margin** of **error**.

E.g. A population of **52 million**, correct to **2 significant figures**, lies somewhere between **51.5 million** and **52.5 million**.
In this population, **51.5** is the **lower bound**.
Another population is **147 million**, correct to **3 significant figures**, lies somewhere between **146.5** and **147.5 million**.
In this population, **146.5** is the **lower bound**.
The **lower bound** for the **difference** between these two populations is **94 million**, i.e. **146.5 – 52.5**.

Lower quartile

When the **total frequency** on a **cumulative frequency curve** (ogive) is divided into **quarters**, the **score** corresponding to the **lower quarter** is the **lower quartile**.

On the cumulative frequency curve below, the **lower quartile**, Q_1, is **3.4**.



Lowest common multiple

The **lowest common multiple (L.C.M.)** is the **lowest** quantity into which each of a group of quantities will **divide exactly**.

E.g. **84** is the **L.C.M.** of **28** and **42**.

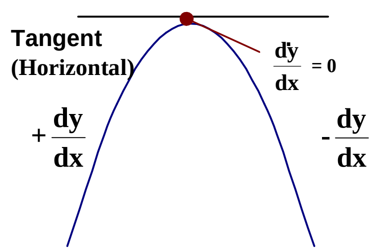
The quickest way to find the **L.C.M.** of a group of numbers is to say the **multiplication** tables for the **bigger** (or biggest) **number**, in this case the '**42-times**' and the **first multiple** of **42** into which **28** divides exactly, i.e. **84**, is the **L.C.M.**

The **L.C.M.** of $\mathbf{a^2b^2c^2}$, $\mathbf{ac^3}$ and $\mathbf{a^3b^4c^4}$ is $\mathbf{a^3b^4c^4}$.

Maximum turning point

The **gradient** of the **tangent** to the curve **before** the turning point is **+**,

at the turning point is **0** and **after** the turning point is **-**.



MAXIMUM TURNING POINT

Maximum value of y = f(x)

If **y = f(x)**, the maximum value of the function is the value of **y** at the **maximum turning point**.

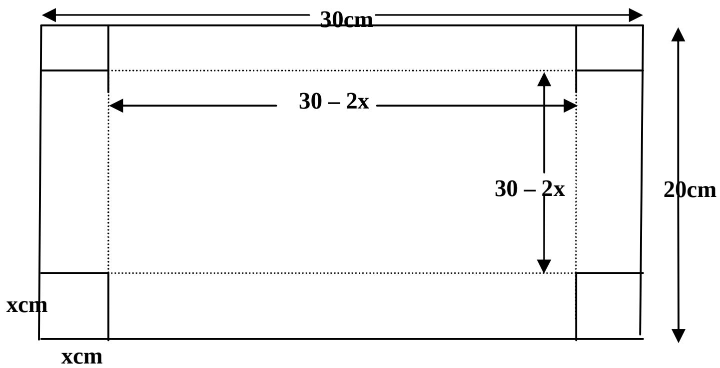
E.g. From a rectangular sheet of cardboard of length **30cm** and breadth **20cm**, an equal square of side **xcm** is cut from each corner, so that the remaining flaps can be folded upwards to form a cuboid.

Find:

(a) the **volume** of the cuboid in terms of **x** and

(b) the **value** of **x** which would give the **maximum** volume. Also, find the maximum volume.

See the diagram on the next page:



Volume of **cuboid**	=	**length ´ breadth ´ height**	
Þ **V**	=	**(30 – 2x) ´ (20 – 2x) ´ x**	
Þ **V**	=	**x(30 – 2x)(20 – 2x)**	
\ **V**	=	$\mathbf{x(600 - 100x + 4x^{2})}$	
Þ **V**	=	$\mathbf{600x - 100x^{2} + 4x^{3}}$ …	(a)
Þ $\mathbf{\frac{dV}{dx}}$	=	$\mathbf{600 - 200x + 12x^{2}}$.	
$\mathbf{\frac{dV}{dx}}$	=	**0**	at turning points
Þ $\mathbf{600 - 200x + 12x^{2}}$	=	**0**	
Þ $\mathbf{150 - 50x + 3x^{2}}$	=	**0**	(÷ 4).
$\mathbf{3x^{2} - 50x + 150}$	=	**0**	

x = $\mathbf{\frac{-b \pm \sqrt{b^{2} - 4ac}}{2a}}$

Þ **x** = $\mathbf{\frac{50 \pm \sqrt{2500 - 1800}}{6}}$

\ **x** = $\mathbf{\frac{50 \pm \sqrt{700}}{6}}$

\ **x** = $\mathbf{\frac{50 \pm 26.46}{6}}$

Þ **x** = **12.74** or **3.92**.

x = **12.74**

Þ **V** = **- 315.56** cm^{3} … (Reject – minus volume.)

x = **3.92**

Þ **V** = **1056.31** cm^{3} … (Accept.)

\ **x = 3.92**cm gives the **maximum volume = 1056.31** $\mathbf{cm^{3}}$.

Mean

The **mean** is the ordinary **arithmetical average**
- simply 'add them all up and divide by the number of them'.

E.g. In a survey, **20** children were asked how many magazines they read in each week.
The results of the survey were as follows:

2	6	0	4	1	3	4	2	0	4
2	4	1	1	7	3	4	5	6	2

When we organize the data, we have:

Score	**Frequency**	**Score × Frequency**
0	2	0
1	3	3
2	4	8
3	2	6
4	5	20
5	1	5
6	2	12
7	1	7
Totals	**20**	**61**

N.B. IIII = 4
but ~~IIII~~ **= 5**

The **mean** number of magazines read per week

= **Total Number of Magazines Read** / **No. of Children**

= $\frac{61}{20}$

= $\frac{6.1}{2}$

= **3.05**

= Therefore, **the mean number of magazines read per week is 3.05**.

(**N.B.** This means that, going on the results of this survey, a total of **305** magazines would be the expected number of magazines read in a week by **100 children**.)

Mean of a grouped distribution

The **table** below gives a **grouped frequency distribution** for the **heights** (to the nearest cm) of **100** fifth-year pupils in a secondary school:

Class Interval	Height (cm)	Frequency	Mid-point of Class Interval
1st	145-149	5	147
2nd	150-154	8	152
3rd	155-159	15	157
4th	160-164	35	162
5th	165-169	20	167
6th	170-174	10	172
7th	175-179	7	177
	TOTAL	**100**	

To find the mean height of a pupil in the grouped frequency shown in the table above, multiply each class-interval's mid-point by its frequency, add these up and divide by 100, i.e. the total number of pupils.
We have:

Frequency f	**Mid-point of Class Interval x**	**Mid-point × Frequency f(x)**
5	**147**	**735**
8	**152**	**1216**
15	**157**	**2355**
35	**162**	**5670**
20	**167**	**3340**
10	**172**	**1720**
7	**177**	**1239**
Σf = 100		**Σf(x) = 16275**

The **mean height** of a pupil is, therefore:

$$\frac{Sf(x)}{Sf} = \frac{16275}{100} = 162.75\text{cm}$$

$= \textbf{163cm}$ (to the nearest cm).

Median

The median is the **middle** score, when the scores are **ordered**, of an **odd** number of scores; when there is an **even** number of scores, the median is the **average** of the **two middle** scores.

E.g. In a survey, **20** children were asked how many magazines they read in each week.
The results of the survey were as follows:

2	6	0	4	1	3	4	2	0	4
2	4	1	1	7	3	4	5	6	2

When the **20 scores** are arranged **in ascending order of size**, we have:
0 0 1 1 1 2 2 2 2 3 3 4 4 4 4 4 5 6 6 7

The **2 middle scores** in the **20** scores are, therefore, the **10th score** and the **11th scores** are **3** and **3**.
Average of 3 and 3 is $\frac{3 + 3}{2}$

$= \mathbf{3}$.

Therefore, the **median number of magazines read per week is 3**.

When the **total frequency** on a **cumulative frequency curve** (ogive) is divided into **quarters**, the **score** corresponding to the **second quarter** (i.e. the ***middle***) is the **median**.

See **Cumulative Frequency diagram** (Ogive) below.

If we divide the **total frequency** into **quarters** the **score** corresponding to the **middle** is the **median**.

E.g. In a survey, **20** children were asked how many hours they spend on sports in each week.

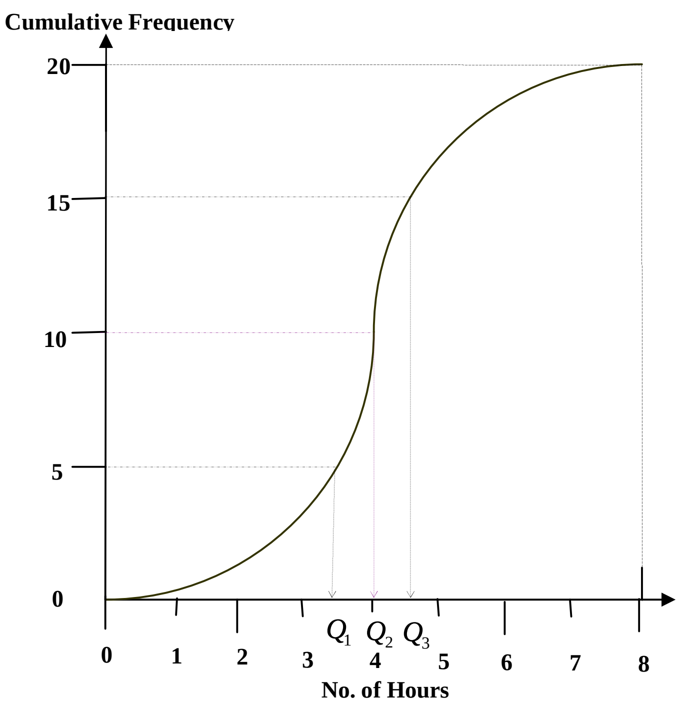
The results of the survey were as follows:

4	**5**	**1**	**4**	**5**
3	**4**	**2**	**6**	**4**
4	**4**	**5**	**5**	**7**
5	**4**	**5**	**8**	**5**

<u>Cumulative Frequency Table from given data</u>

Number of Hours (less than or equal to)	**Cumulative Frequency**
0	**0**
1	**1**
2	**2**
3	**3**
4	**10**
5	**17**
6	**18**
7	**19**
8	**20**

The **cumulative frequency diagram** (ogive) compiled from the above data looks like this:



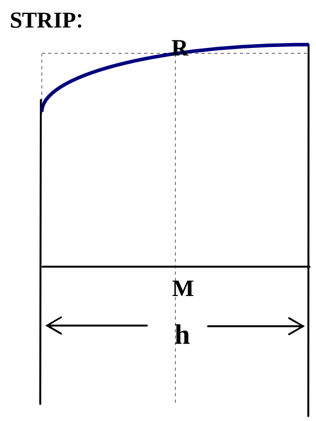
The **median**, Q_2 = **4** hours.

Mid – Ordinate Rule

Divide the area up into a number of strips, each of width **h.** Using this rule, the strips are approximated to **rectangles**, with **lengths** equal to the **lengths** of the **mid-ordinates** and **breadths** equal to **h**.

Again, the **smaller** the value of **h** chosen, the **closer** the strip will be to the **area** of the **rectangle** formed using the **mid-ordinate**.

The 'picture' looks like this:



M is the **mid-point** of **h** and **RM** is the **mid-ordinate**. The **area** under the **curve** is regarded as being **equal** to the **area** of the **rectangle**, i.e.

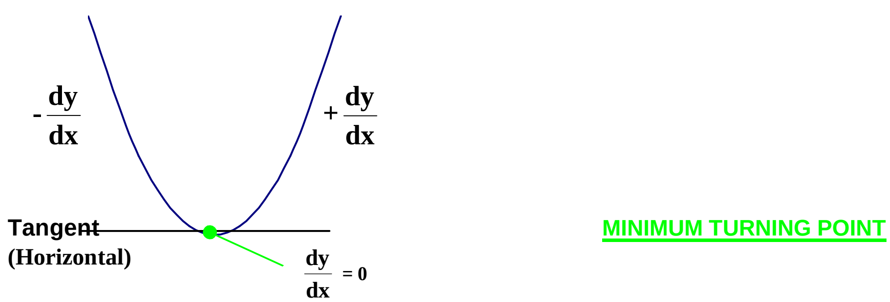
RM ´ h, using **The Mid-Ordinate Rule**.

The total area can be factorised to give the formula for this rule:
$\mathbf{h\,(y_1 + y_2 + y_3 + y_4 + y_5)}$.

Minimum turning point

The **gradient** of the **tangent** to the curve **before** the turning point is **-**,

at the turning point is **0** and **after** the turning point is **+**.



Minimum value of y = f(x)

If **y = f(x)**, the minimum value of the function is the value of **y** at the **minimum turning point**.

E.g. Find the minimum value of $\mathbf{y = x^2 + x - 2}$.

(i) $\mathbf{\frac{dy}{dx} = 2x + 1}$.

(ii) $\mathbf{2x + 1 = 0}$.

(iii) $\mathbf{2x = -1}$ Þ $\mathbf{x = ½}$.

\ **(½, 1¼)** is the **minimum turning point** on the curve,

$y = \mathbf{x^2 + x - 2}$.

The minimum value of **y** is, therefore, **1¼**.

Mode

The mode can be thought of as the 'fashionable' score.
(***Mode*** is French for *fashion*.)
This means that it is the score with the **highest frequency**,
(i.e. it occurs the **most often**.)

E.g. In a survey, **20** children were asked how many magazines they read in each week.
The results of the survey were as follows:

2	6	0	4	1	3	4	2	0	4
2	4	1	1	7	3	4	5	6	2

When the **20 scores** are arranged in order, we have:
0 0 1 1 1 2 2 2 2 3 3 4 4 4 4 4 5 6 6 7
The score that occurs **most often** is **4**.

Therefore, **4** is the **modal number of magazines read per week** per pupil.

Moment

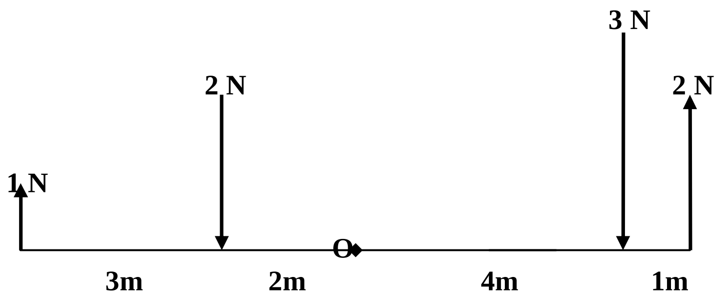
(Mechanics – GCSE Additional and Advanced Subsidiary)

A moment is the **turning** effect of a force applied at a **point**, causing **rotational motion**. The moment of a force about a point is:
magnitude of **force** × **perpendicular distance** of force from the **pivot**.

For a body to be in **equilibrium**, the **resultant force** acting on the body must be **zero** and the **resultant moment** is **zero**.

If the force is measured in newtons and the distance in metres, the **moment** of the force is measured in newton metres, **N m**.
Moments can be **clockwise** or **anti-clockwise** and should always have their sense clearly stated; a moment has **magnitude** and **direction**.

When finding the **resultant moment** of two or more forces about a point, one direction is taken as **positive** and the other **negative**.



Example:

Find the resultant moment about the point **O** of the forces shown in the diagram above.

Method:

Taking clockwise as + and – as anti-clockwise, we have:
$1 \times 5 - 2 \times 2 + 3 \times 4 - 2 \times 5 \quad = \quad 5 - 4 + 12 - 10 \quad = \quad 3$ Nm

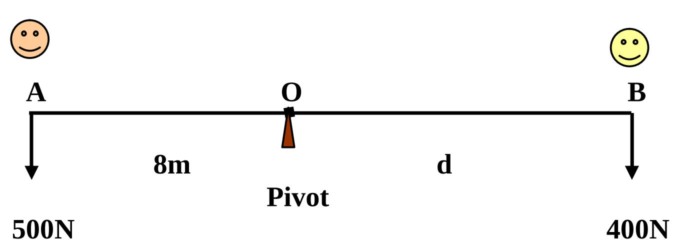
$\Rightarrow$ **3 N m clockwise**.

For a body to be in equilibrium, the **resultant force** acting on the body must be **zero** and the **resultant moment** is **zero**.

Example:

Two children, **A** and **B**, whose weights are **500N** and **400N** respectively, are balanced on a see-saw, whose pivot is at **O**.

If **A** is positioned **8m** from the pivot, how far from the pivot is **B**?



Taking moments around **O**, clockwise being **+**, we have:

$$400d - 500 \times 8 = 0$$
$$\Rightarrow \quad 400d - 4000 = 0$$
$$\Rightarrow \quad d = 10$$

$\therefore$ **B** is **10m** from the pivot for equilibrium.

Momentum and Impulse (Mechanics – GCSE Additional and Advanced Subsidiary)

The **momentum** of a body is: **Mass ´ Velocity**.

If the units of mass and velocity are **kg** and **m/s** respectively,
then the **units** of **momentum** are **newton-seconds** (**N s**).

Since the momentum of a body depends upon its velocity,

momentum is a vector quantity.

The **impulse** of a constant force **F** is **F ´ t**,
where **t** is the time for which the force is acting.

F = ma gives:		$Ft = ma \times t$
v = u + at gives:		$Ft = m(v - u)$
	$\therefore$	$Ft = mv - mu$

Therefore: **Impulse = Change in momentum**.

Example 1:

Find the **magnitude** of the **momentum** for each of the following:

(i) A lorry of mass10 tonnes moving with a speed of 25m/s.
(ii) A ball of mass 250g moving with a speed of 12m/s.
(iii) A girl of mass 55kg moving with a speed of 3m/s.

Method:

(i)	Momentum	=	mass $\times$ velocity
		=	$(10 \times 1000) \times 25$
		=	**250000 N s**.
(ii)	Momentum	=	mass $\times$ velocity
		=	$(250 \div 1000) \times 12$
		=	**3 N s**.
(iii)	Momentum	=	mass $\times$ velocity
		=	55×3
		=	**165 N s**.

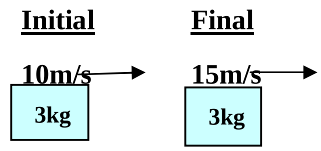
If the **velocity** of a body changes from **u** to **v**, then its momentum also changes.
The initial momentum is **mu** and the final momentum is **mv**.

Example 2:

Find the **change** in **momentum** of a body of mass 3kg when its speed changes:

(i) from 10m/s to 15m/s in the same direction.
(ii) from 8m/s to 5m/s in opposite directions.

Method: Draw 2 diagrams:

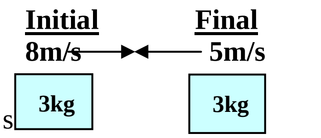


Taking velocities to right as +:

(i)
Initial Momentum = $3 \times 10 = 30$ N s

Final Momentum = $3 \times 15 = 45$ N s

$\therefore$ Change in momentum = **15 N s**.



Taking velocities to right as +:
(ii)
Initial Momentum = $3 \times 8 = 24$ N s

Final Momentum = $3 \times (-5) = -15$ N s

$\therefore$ Change in momentum = **39 N s**.

Impulse = Change in momentum.

When **two particles collide**, each receives an impulse from the other **equal** in **magnitude** but **opposite** in **sign**. The sum of the momenta before impact is equal to the sum of the momenta after impact.
(*The principle of Conservation of Linear Momentum.*)

The **total change** in momentum is, therefore, **zero**.

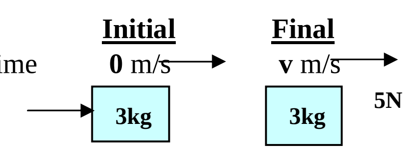
Example 1:

A body of mass **3kg** is initially at rest on a smooth horizontal surface.
A horizontal force of **5N** acts on the body for **8** seconds.

Find: (i) the magnitude of the impulse given to the body.
(ii) the magnitude of the final momentum of the body.
(iii) the final speed of the body.

Method:

(i) Draw 2 diagrams:
Impulse = Force $\times$ Time
$= 5 \times 8 =$ **40 N s**.



(ii) Initial Momentum, mu = 0

Impulse = mv – 0

$\therefore$ 40 = mv, i.e. final momentum.

(iii) mv = 40
$\therefore$ 3v = 40
$\therefore$ v = $\mathbf{13\frac{1}{3}}$, i.e. final speed.

Moving Averages

Data reported at regular intervals of time tend to have an ***underlying trend***. This type of data can be 'smoothed out' using moving averages to form a ***trend line***, which can be extrapolated to predict future values.

Classifying Variations

(i) **Secular** trend is found when the direction of the data keeps going **upwards** or **downwards** over a long period of time:
E.g.1 The high jump record keeps **increasing** over a long period, giving an **upwards – moving secular trend** in the data.
E.g.2 The winning time in a marathon keeps **decreasing** over a long time period, thereby giving a **downwards – moving secular trend** in the data.

(ii) **Seasonal** variation occurs when the data follow a pattern during corresponding months in successive years, for example heating bills. Electricity and heating bills fluctuate with the seasons **-** higher bills in the colder weather, lower bills in the warmer weather.

(iii) **Cyclical** variations occur when long periods of time follow the trend line, for example several years of prosperity in the economy followed by several years of recession, forming a pattern over time.

(iv) **Random** variations occur when unpredictable events like a war or a 'crash' in the stock markets happens. These variations **cannot** be 'smoothed out' using moving averages, since they are **irregular**.

Construction of the trend line using moving averages

E.g. The table below shows the number of properties let by Nuhomes Estate Agency over the past three years.

	1st quarter	**2nd quarter**	**3rd quarter**	**4th quarter**
2001	328	317	335	351
2002	355	343	361	379
2003	381	371	388	406

(i) Plot these data on a graph.

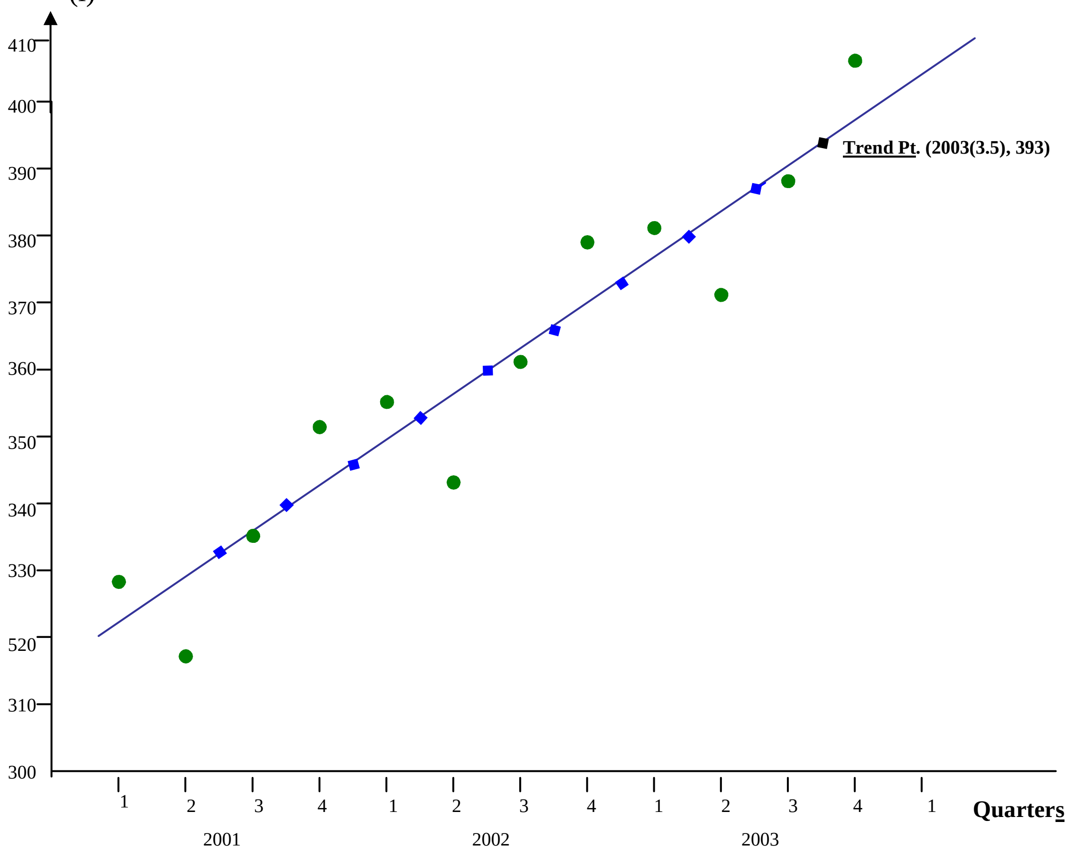
(ii) Calculate appropriate moving averages to smooth the data.

(iii) Plot these averages on the graph and draw the trend line. *Showing clearly* where any reading is taken use the trend line to estimate how many properties are likely to be let in the first quarter of 2004.

(iv) Why do we use moving averages? (C.C.E.A. Additional – 2004)

Method:

(i)



N.B.

The **positioning** of the **moving average** is very important. **The moving average** must be plotted at the **mid-point** of the data from which it is calculated.

The **1st moving average** is computed from quarters **1**, **2**, **3** and **4** of **2001**, to give the **1st moving average** – it lies **mid – way** between quarters **2** and **3** of **2001**.

The **2nd moving average** lies **mid – way** between quarters **2** and **3** of **2002** and so on.

Using our trend line, the '**trend point**', the **average** taken from quarters **2**, **3** and **4** of **2003** and quarter **1** of **2004**, gives **<u>393 properties</u>**.

Working backwards, (**371 + 388 + 406 + x**) ÷ **4 = 393** gives **x = 407**, i.e. **<u>407 properties</u> estimated for 1st quarter of 2004**.

<u>4 – point moving averages</u>

328		
317	332.75	328 + 317 +335 + 351 = 1331 ÷ 4 = 332.75
335	339.5	317 +335 + 351 + 355 = 1358 ÷ 4 = 339.5
351	346	335 + 351 + 355 + 343= 1384 ÷ 4 = 346
355	352.5	351 + 355 + 343 + 361 = 1410 ÷ 4 = 352.5
343	359.5	355 + 343 + 361 + 379 = 1438 ÷ 4 = 359.5
361	366	343 + 361 + 379 + 381 = 1464 ÷ 4 = 366
379	373	361 + 379 + 381 + 371 = 1492 ÷ 4 = 373
381	379.75	379 + 381 + 371 + 388 = 1519 ÷ 4 = 379.75
371	386.5	381 + 371 + 388 + 406 = 1546 ÷ 4 = 386.5
388		
406		

(iii) $$\frac{\mathbf{371 + 388 + 406 + x}}{\mathbf{4}} = \mathbf{393}$$

$$\mathbf{1165 + x = 1572}$$
$$\mathbf{x = 407}.$$

This means that our **estimate** of the number of properties likely to be let in the 1^{st} **quarter of 2004** is **407**.

(iv) We use moving averages to ‘smooth’ data so that we can try to predict future values.

Multiple

A **multiple** is a quantity into which another quantity **can be divided** without leaving a remainder. It is useful to think of the **multiplication** tables as **multiples**.
E.g. The **‘2-times’** tables give **multiples** of **2**, namely **2**, **4**, **6**, **8**, …, the **‘3-times’** tables give **multiples** of **3**, namely **3**, **6**, **9**, **12**, …, etc.

Newton’s Laws of Motion (Mechanics – GCSE Additional and Advanced Subsidiary)

<u>First Law</u> – a **change** in the **state of motion** of a body is caused by a **force**.

This means that a body will stay in a state of **rest** or in **constant motion** *unless* it is acted upon by an outside force.
If **forces** act on a body and it remains at **rest**, the forces must balance; hence the **resultant** force in any direction must be **zero**.
A body in motion can change its velocity or direction ***only if*** a resultant force acts upon it.
A **force**, therefore, is a **vector quantity** that causes a **change** in the **state of motion** of a body.

The **unit** of force is the **newton** (**N**).

A force of 1**N** produces an acceleration of 1$\mathbf{m/s^2}$ in a body of mass 1**kg**.
The **weight** of a body is the force exerted upon it by **gravity** g ($g = 9.81\ m/s^2$).

Generally, the **weight** of a body of **mass m kg** is **mg N**.

E.g. A person with a ***mass*** of **60 kg** has a ***weight*** of approximately **600 N** (g is often taken as 10 m/s^2).

Second Law – a **resultant force** acting on a body causes **acceleration.**

The acceleration is **proportional** to the **force** and the same force will **not** cause the same acceleration in all bodies; the acceleration depends on the **mass** of the body on which the force acts.
The standard **unit** of **mass** is the **kilogram** (kg).
A **force** of **1N** produces an **acceleration** of $\mathbf{1m/s^2}$ in a **mass** of **1kg**.

Generally, a force of **F** newtons acting on a body of mass **m** kg produces an acceleration of **a** m/s^2, giving the **equation** of **motion**:

$$F = ma.$$

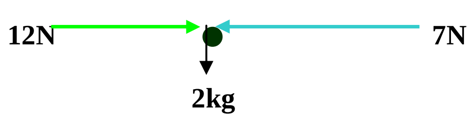
E.g. 1. A **force** of **5N** acting on a body of **mass 10kg** has an **acceleration** of $\mathbf{0.5m/s^2}$:

$$5 = 10a$$
$$\Rightarrow a = 0.5.$$

E.g. 2. The **resultant force** that would give a body of **mass 250g** an **acceleration** of $\mathbf{12m/s^2}$ is **3N**:

$$F = 0.25(12)$$
$$\Rightarrow F = 3N.$$

E.g. 3. A body of **mass 2kg** rests on a smooth horizontal surface. Horizontal forces of **12N** and **7N** start to act on the particle in opposite directions.
Find the **acceleration** of the body.

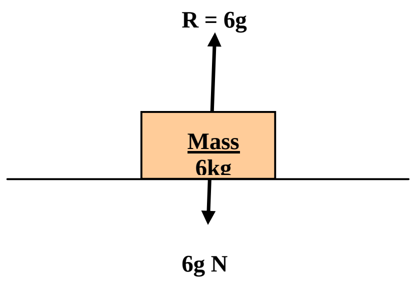


$$F = ma$$
$$\Rightarrow 12 - 7 = 2a$$
$$\Rightarrow 5 = 2a$$
$$\Rightarrow 2.5 = a$$
$\Rightarrow$ acceleration of $\mathbf{2.5m/s^2}$.

Third Law – **Action** and **Reaction** are **equal** and **opposite**

If two bodies **P** and **Q** are in contact and exert **forces** on each other, the **forces** are **equal** in **magnitude** and **opposite** in **direction**.

E.g. A case with a **mass** of **6kg** rests on a horizontal table. The case exerts a force on the table and the table 'reacts' by exerting an equal and opposite force on the case. Since the case is at rest, the reaction force **R** is **6g**, i.e. the weight of the case.



Non-terminating repeating decimal number

A **non-terminating repeating decimal** number is a **rational** number,

e.g. $0.\dot{3} = 0.3333\ldots = \frac{1}{3}$

$0.\dot{1}\dot{8} = 0.1818\ldots = \frac{2}{11}$

$0.0\dot{1}\dot{5} = 0.01515\ldots = \frac{1}{66}$.

Normal

A **normal** is a line **perpendicular** to the **tangent** at the point of tangency.

The **product** of their **gradients** is, therefore, **-1**.

E.g. If a tangent has gradient **– 2**, the normal has gradient $\frac{1}{2}$;

if a tangent has gradient $-\frac{5}{3}$, the normal has gradient $\frac{3}{5}$.

Worked Example:

If $y = 2x^2 + x - 1$, find the **gradient** of the curve at the point where **x = 1**, and, *hence*, find the **equation** of the **normal** to the curve at the point **(1, 2)**.

$y = 2x^2 + x - 1$

Þ $\frac{dy}{dx} = 4x + 1$

$x = 1$ Þ $\frac{dy}{dx} = 4(1) + 1 = 5$.

\ the **gradient** of the **tangent** at the point where **x = 1** is **5**.

Q the **tangent** has **gradient 5**, the **normal** has **gradient** $-\frac{1}{5}$.

Then **y** = **mx** + **c** gives:

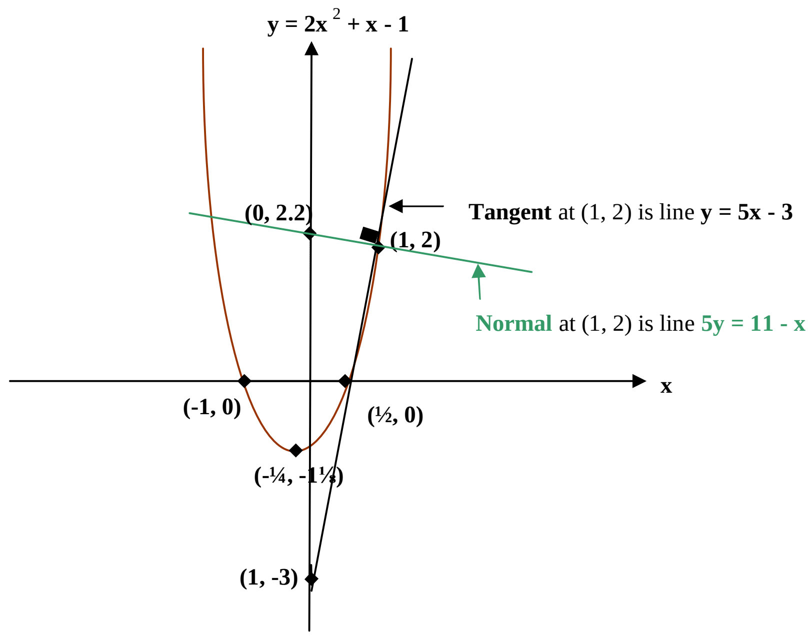
2 = $-\frac{1}{5}(1) + c$

Þ **c** = $\frac{11}{5}$.

\ **y** = $-\frac{1}{5}x + \frac{11}{5}$

or **5y** = **11 – x** is the **equation** of the **normal** at the point **(1, 2)**.

` See the diagram below:



Ogive

An ogive is a **cumulative frequency curve**.

See **Cumulative Frequency diagram** (Ogive) below.

If we divide the **total frequency** into **quarters**, the **score** corresponding to the **lower quarter** is the **lower quartile**, the **score** corresponding to the **middle** is the **median** and the **score** corresponding to the **upper quarter** is the **upper quartile**.
The **interquartile range** is the **difference** between the **score readings** provided by the **upper** and **lower quartiles**.

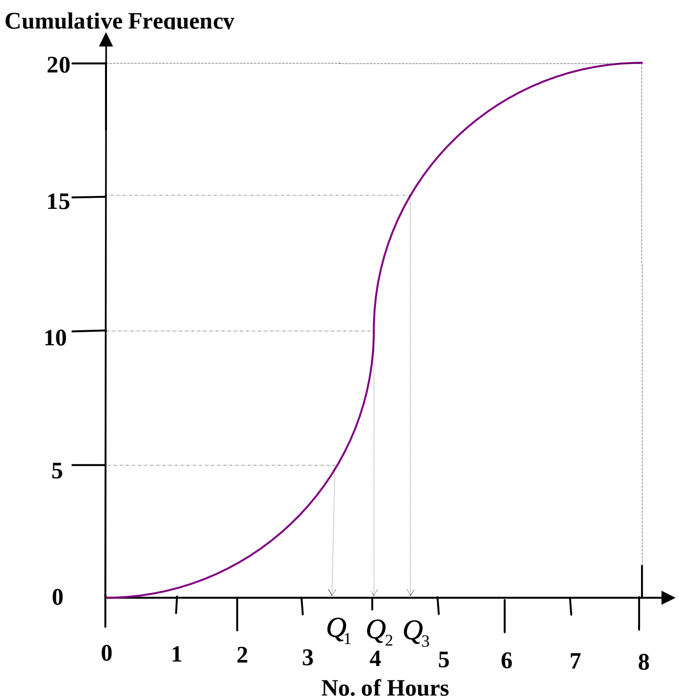
E.g. In a survey, **20** children were asked how many hours they spend on sports in each week.
The results of the survey were as follows:

4	**5**	**1**	**4**	**5**
3	**4**	**2**	**6**	**4**
4	**4**	**5**	**5**	**7**
5	**4**	**5**	**8**	**5**

Cumulative Frequency Table from given data

Number of Hours (less than or equal to)	Cumulative Frequency
0	0
1	1
2	2
3	3
4	10
5	17
6	18
7	19
8	20

The **cumulative frequency diagram** (ogive) compiled from the above data looks like this:



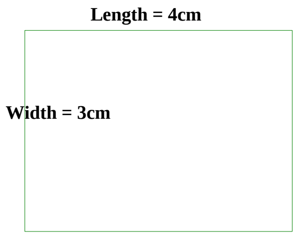
The **lower quartile**, Q_1 = **3.4**; the **median**, Q_2 = **4**;
the **upper quartile**, Q_3 = **4.6**.

This gives: **Median** no. of hours = **4**.
Interquartile range = 4.6 – 3.4 = 1.2 hours.

Perimeter

The **perimeter** of a closed shape is the distance around its **outline**, like the fence that encloses a garden.

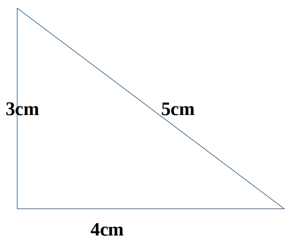
(a) **Rectangle**



Perimeter = 2 lengths + 2widths
or **2 × (length + width)**

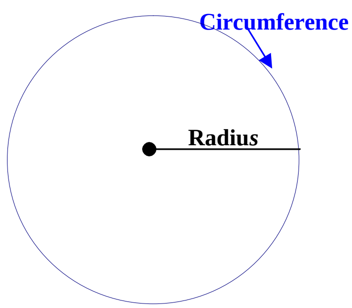
The rectangle shown has perimeter
2 × (4cm + 3cm) = 2 × 7cm = 14cm.

(b) **Triangle**



The perimeter of the triangle shown is:
3cm + 4cm + 5cm = 12cm.

(c) **Circle**



The perimeter of a circle has a special name called the circumference. The circumference is $2 \times \pi \times$ radius, where π is approximately $3\frac{1}{7}$, **3.14** or **3.142.**

The circumference of the circle shown, if we take π as 3.142 is:
$2 \times 3.142 \times 2.5$ = 15.71cm.

Pie chart

E.g. In a survey, **20** children were asked how many magazines they read in each week.
The results of the survey were as follows:

2	6	0	4	1
3	4	2	0	4
2	4	1	1	7
3	4	5	6	2

The results of this survey on magazines could be shown on a pie chart. The **pie chart is a circle divided into a number of sectors**, each of which displays **a proportion of the whole sample**.
The **size of the angle in each sector must be determined** before a pie chart can be drawn. Remember that there are **360° in the whole circle**. In our survey, we have **20 children altogether**.

Therefore, **20 children = 360°**
i.e. **1 child = 18°**.

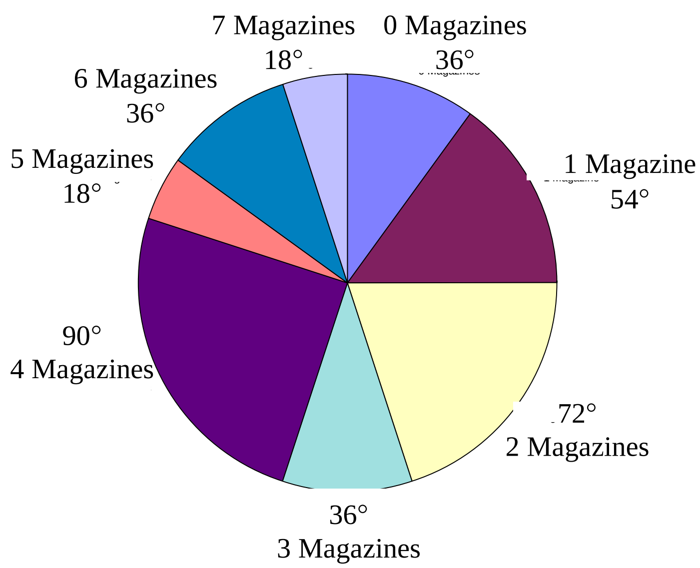
Next, we **change the numbers of children to degrees**, as follows:

No. of Magazines	No. of Children	Degrees
0	**2**	**2 ´ 18° = 36°**
1	**3**	**3 ´ 18° = 54°**
2	**4**	**4 ´ 18° = 72°**
3	**2**	**2 ´ 18° = 36°**
4	**5**	**5 ´ 18° = 90°**
5	**1**	**1 ´ 18° = 18°**
6	**2**	**2 ´ 18° = 36°**
7	**1**	**1 ´ 18° = 18°**
TOTALS	**20**	**360°**

Now we are ready to do the pie chart.

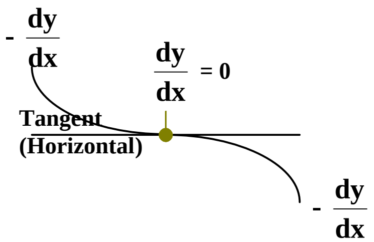
We use a **protractor to measure the angle** which we need to draw in **each sector**.

PIE CHART REPRESENTING RESULTS OF SURVEY ON MAGAZINES



Point of inflexion (GCSE Additional - Pure and Advanced Subsidiary - Pure)

1. The **gradient** of the **tangent** to the curve **before** the turning point is **-**, **at** the turning point is **0** and **after** the turning point is **-**.



POINTS OF INFLEXION

Diagram (a)

2. The **gradient** of the **tangent** to the curve **before** the turning point is **+**, **at** the turning point is **0** and **after** the turning point is **+**.

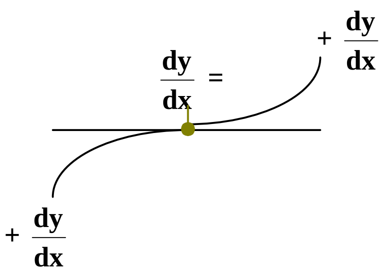


Diagram (b)

Powers

x can be raised to a **power** which means it is **multiplied by itself** the number of times that the power states:

$\mathbf{x^2}$ means $\mathbf{x \times x}$.

Note the **difference** between $\mathbf{x^2}$ and **2x**: $\mathbf{x^2 = x \times x}$ and $\mathbf{2x = x + x}$.

If $\mathbf{x = 4}$, then $\mathbf{x^2 = 4 \times 4 = 16}$ and $\mathbf{2x = 2 \times 4 = 8}$.

Again, note the **difference** between $\mathbf{x^3}$ and **3x**: $\mathbf{x^3 = x \times x \times x}$ and $\mathbf{3x = 3 \times x}$.

If $\mathbf{x = 4}$, then $\mathbf{x^3 = 4 \times 4 \times 4 = 64}$ and $\mathbf{3x = 3 \times 4 = 12}$.

Polygon

A **polygon** is literally a '**many-sided shape**', derived from the Greek word ***poly*** meaning '**many'**.It is **any closed plane figure**, bounded on **all sides** by **straight lines**. It is obvious, then, that the shape must have at least three sides to fulfil the required criteria to be called a 'polygon'.

A number of the earlier polygons have **special names**:

TRIANGLE (3-sided polygon)	**(i) Sum of interior angles = 180°** **(ii) Sum of exterior angles = 360°**
QUADRILATERAL (4-sided polygon)	**(i) Sum of interior angles = 360°** **(ii) Sum of exterior angles = 360°**
PENTAGON (5-sided polygon)	**(i) Sum of interior angles = 540°** **(ii) Sum of exterior angles = 360°**

<u>HEXAGON</u>	**(i) Sum of interior angles = 720°**
(6-sided polygon)	**(ii) Sum of exterior angles = 360°**
<u>HEPTAGON</u>	**(i) Sum of interior angles = 900°**
(7-sided polygon)	**(ii) Sum of exterior angles = 360°**
<u>OCTAGON</u>	**(i) Sum of interior angles = 1080°**
(8-sided polygon)	**(ii) Sum of exterior angles = 360°**

NOTE: Some more polygons have special names; however, knowledge of these is not required generally.

(i) <u>Interior Angles</u>

When a polygon is divided up into **triangles** the **number** of **triangles** is always **two less** than the **number** of **sides** on the polygon:

3 sides ⇒	**1** triangle	=	**1 ´ 180°**	=	**180°**
4 sides ⇒	**2** triangles	=	**2 ´ 180°**	=	**360°**
5 sides ⇒	**3** triangles	=	**3 ´ 180°**	=	**540°**

.
.
.

n sides ⇒ **(n - 2)** triangles = **180(n – 2)**°.

Therefore, the **sum** of the **interior angles** of **any polygon** is **180(n – 2)**°, where **n** is the **number of sides** on the polygon.

(ii) <u>Exterior Angles</u>

The **sum** of the **exterior angles** of **any polygon** is **360°**, **regardless** of the **number** of **sides** on the polygon.

N.B. The interior angle added to the exterior angle is always 180°.

<u>Regular Polygons</u>

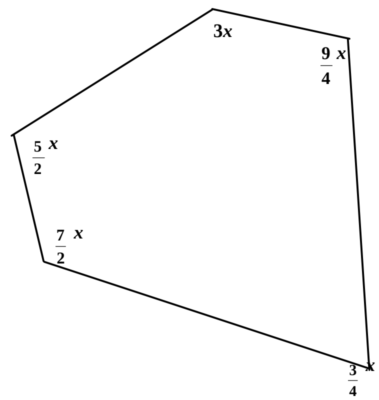
A **regular** polygon is a polygon whose **sides** are **all equal**.
It follows that all **interior angles** are **equal** to each other and all **exterior angles** are **equal** to each other.

(i) The size of **each interior angle** in a regular **n**-sided polygon is $\frac{180(n - 2)°}{n}$.

(ii) The size of **each exterior angle** in a regular **n**-sided polygon is $\frac{360°}{n}$.

Worked Examples on Polygons

Q.1. In the pentagon below, find the value of ***x***°, and hence, the size of each of the interior angles.



Sum of **interior angles** in a pentagon is:

180(5 – 2)° = 540° **(i.e. 3 triangles)**

Then we have:

$$3x + \frac{9}{4}x + \frac{3}{4}x + \frac{7}{2}x + \frac{5}{2}x = 540$$

Þ $12x = 540$

\ $x = 45$.

\ ÐA = 135°

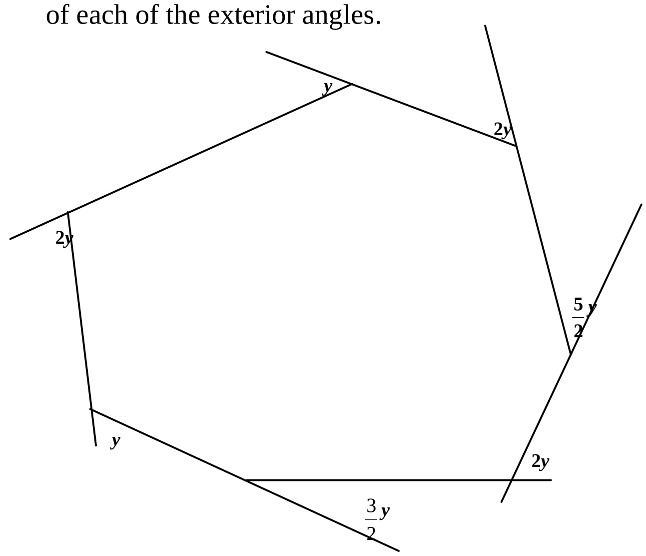
ÐB = 101.25° ÐC = 33.75°

ÐD = 157.5°

ÐE = 112.5°,

giving the total of **540°**.

Q.2. In the heptagon below, find the value of **y**, and hence, the size of each of the exterior angles.



Sum of **exterior angles** is **360°**.

Then we have:

$$\boldsymbol{y + 2y + \tfrac{5}{2}y + 2y + \tfrac{3}{2}y + y + 2y = 360}$$

Þ $\boldsymbol{12y = 360}$

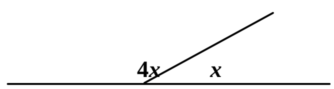
\ $\boldsymbol{y = 30}$.

\ **ÐNAG = 30°**
ÐMBA = 60°
ÐLCB = 75°
ÐKDC = 60°
ÐJED = 45°
ÐHFE = 30°
ÐPGF = 60°,

giving the total of **360°**.

Q.3. In a regular polygon, each interior angle is four times the size of each exterior angle.
Find the number of sides in the polygon.

We have:



If we let ***x*** be the **exterior** angle,
then **4*x*** is the **interior** angle.

Q $\boldsymbol{4x + x = 180°}$ (i.e. a straight line)
Þ $\boldsymbol{5x = 180°}$
\ $\boldsymbol{x = 36°}$.

We have now found the **exterior angle** to be **36°**.

Since exterior angles in a **regular** polygon are **all equal**, and add up to **360°**, this polygon must have:

$$\boldsymbol{\frac{360}{36}} = \textbf{10 sides}.$$

Prime number

A **prime number** has **no factors except itself** and **1**.
This means that **no other number will divide into a prime number** without leaving a remainder.

E.g. 6 is **not** a prime number because it has **factors** of **3** and **2.**
On the other hand, **31 is a prime number** because no number except **31** and **1** will divide into it without leaving a remainder.

Prime Numbers = {2, 3, 5, 7, 11, 13, 17, 19, . . .}

To find the **H.C.F**. of a set of numbers, write each of the numbers as a **product** of **prime factors** first and then select the **prime factor** (or factors) **common** to all of the numbers.

E.g. Find the **H.C.F**. of **12, 18** and **24**.

12 = **2** × **2** × **3** in prime factors.
18 = **2** × **3** × **3** in prime factors.
24 = **2** × **2** × **2** × **3** in prime factors.

Therefore **2 × 3 = 6** is common to all of the numbers, giving **6** as the **H.C.F**. of **12, 18** and **24**.

Probability

Probability is the law of **chance**.
Probability 0 means **no chance**.
Probability 1 means **complete certainty**.

Since the probability of an event taking place **added** to the probability of it **not** taking place is **1**, the **probability** of an event taking place is the same as (**1 –** the **probability** of that event ***not*** taking place).

E.g. A box contains **10** coloured counters, **4 red**, **3 blue**, **2 green** and **1 yellow**.
One counter is picked at random from the box.
The probability that it is (i) **red** is $\frac{4}{10}$; (ii) **blue** is $\frac{3}{10}$;
(iii) **green** is $\frac{2}{10}$; (iv) **yellow** is $\frac{1}{10}$; (v) **black** is $\frac{0}{10} = 0$.
(**No black** counters in box.) (vi) **not red** is $\frac{6}{10}$; (ii) **not blue** is $\frac{7}{10}$;
(iii) **not green** is $\frac{8}{10}$; (iv) **not yellow** is $\frac{9}{10}$.

Probability laws

'AND' law**:** **MULTIPLY** probabilities.
'OR' law**:** **ADD** probabilities.

This **mnemonic** may be helpful in remembering these laws:
ANDy **MULTIPLIED**. **OR**la **ADD**ed.

E.g. A box contains **10** coloured counters, **4 red**, **3 blue**, **2 green** and **1 yellow**.
Two counters are picked at random from the box (without replacement).

(a) The probability that **both** counters are:

(i) **red** is $\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$;

(ii) **blue** are $\frac{3}{10} \times \frac{2}{9} = \frac{1}{15}$;

(iii) **green** is $\frac{2}{10} \times \frac{1}{9} = \frac{1}{45}$;

(iv) **yellow** is $\frac{1}{10} \times \frac{0}{9} = 0$; (**Only one yellow** counter in box.)

(v) **black** is $\frac{0}{10} \times \frac{0}{9} = 0$. (**No black** counters in box.)

(b) The probability that **only one** of the counters is:

(i) **red** is $\frac{4}{10} \times \frac{6}{9} = \frac{4}{15}$ (i.e. 1^{st} red and 2^{nd} not red);

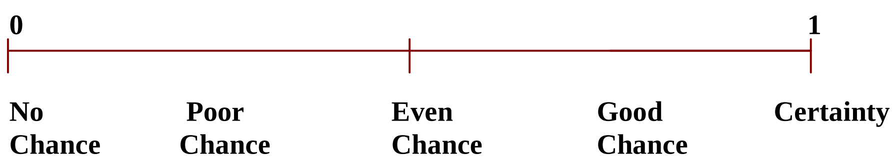
(ii) **blue** are $\frac{3}{10} \times \frac{7}{9} = \frac{7}{30}$ (i.e. 1^{st} blue and 2^{nd} not blue);

(iii) **green** is $\frac{2}{10} \times \frac{8}{9} = \frac{8}{45}$ (i.e. 1^{st} green and 2^{nd} not green);

(iv) **yellow** is $\frac{1}{10} \times \frac{9}{9} = 0$; (**Only one yellow** counter in box.)

(v) **black** is $\frac{0}{10} \times \frac{0}{9} = 0$. (**No black** counters in box.)

Probability Scale



The probability of an event taking place **added** to the probability of it **not** taking place is always **1**.

Probability tree diagram

A **tree diagram** may be drawn to represent probabilities when two (or more) **possible outcomes** have to be considered.

E.g. A box contains **10** coloured counters, **4 red**, **3 blue**, **2 green** and **1 yellow**.

Two counters are picked at random from the box (without replacement).

The probability that:

(i) they are **both red** is $\frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$;

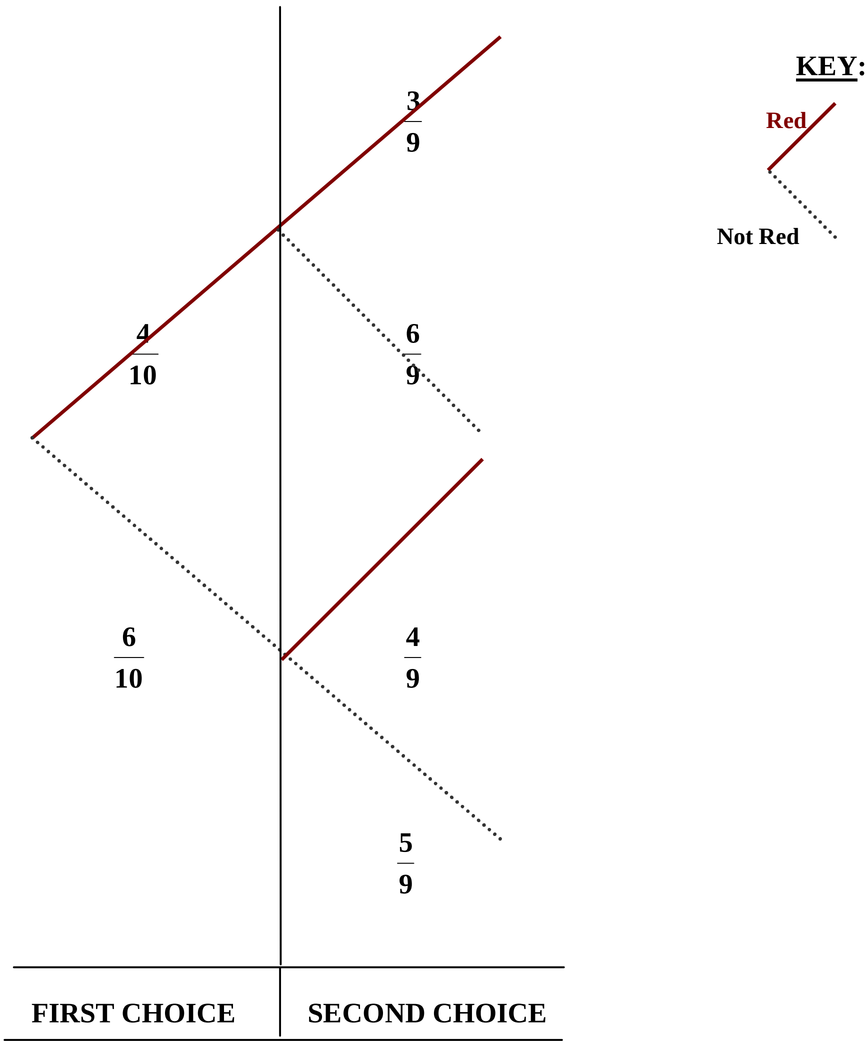
(ii) **only 1** is **red** is $\left(\frac{4}{10} \times \frac{6}{9}\right) + \left(\frac{6}{10} \times \frac{4}{9}\right) = \frac{24}{90} + \frac{24}{90} = \frac{48}{90} = \frac{8}{15}$;

(iii) **neither** is **red** is $\frac{6}{10} \times \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$ (1st **not** red and 2nd **not** red;

(iv) there is **at least 1 red** is **1 -** $\frac{1}{3} = \frac{2}{3}$ (**1** – probability **no red**)

The information obtained in the previous example could be represented on a **tree diagram**, since only two possible outcomes are considered **red** and **not red**. (Notice that '**not red'** may be treated as a colour in its own right.)

<u>Tree Diagram</u>



N.B. Each time the two 'branches' on the tree **add** up to **1**.

Proportion (or **Variation**)

There are **two types** of **proportion** (or **variation**):

(i) **Direct**

(ii) **Inverse**

(i) **Direct Variation**

'**y** is **directly proportional** to **x**' translates to $y = kx$

'p is **directly proportional** to q^2' translates to $p = kq^2$,

and so on.

It is usually possible to find the value of **k**
from information given in the problem.

Below is an **algorithm** which, if followed, makes variation problems easy.

I use the term **'key equation'** to describe the **variation formula**.

E.g. 1. If **y** is **directly proportional** to x^2 and $y = 3$ when $x = 4$, find **x** when $y = 5$.

ALGORITHM

Step 1 ***Write 'key equation':*** $y = kx^2$

Step 2 ***Substitute x = 4, y = 3 into key equation:*** $3 = 16k$

Step 3 ***Solve for k:*** $\frac{3}{16} = k$.

Step 4 ***Put 'found' value of k into key equation:*** $y = \frac{3}{16}x^2$.

Step 5 ***Find x when y = 5:*** $5 = \frac{3}{16}x^2$

Þ $x = 5.16$.

(ii) **Inverse Variation**

'**y** is **inversely proportional** to **x**' translates to $y = \frac{k}{x}$

'**p** is **inversely proportional** to q^2' translates to $p = \frac{k}{q^2}$,

and so on.

The **algorithm** for **inverse variation** problems is the same as for direct variation, the **only difference** being in the **key equations**.

E.g. 2 If **y** is **inversely proportional** to $\mathbf{x}^2$ and **y = 4** when **x = 5**, find **y** when **x = 6**.

ALGORITHM

Step 1 ***Write 'key equation':*** $\quad \mathbf{y} = \frac{k}{x^2}$

Step 2 ***Substitute x = 5, y = 4 into key equation:*** $\quad \mathbf{4} = \frac{k}{25}$

Step 3 ***Solve for k:*** $\quad \mathbf{100} = \mathbf{k}$.

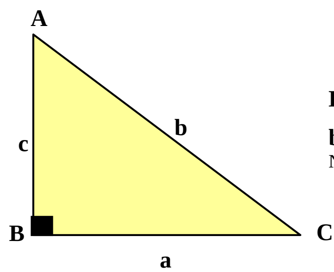
Step 4 ***Put 'found' value of k into key equation:*** $\quad \mathbf{y} = \frac{\mathbf{100}}{\mathbf{x^2}}$

Step 5 ***Find y when x = 6:*** $\quad \mathbf{y} = \frac{100}{6^2}$

$\Rightarrow \quad \mathbf{y} = \mathbf{2}\frac{7}{9}$.

Pythagoras's Theorem

In a **right – angled triangle**, the **side opposite** to the **right angle** is called the **hypotenuse**.
Pythagoras's Theorem states that the **square** on the **hypotenuse** is **equal** to the **sum** of the **squares** on the **other two sides**.



By Pythagoras's Theorem:
$\mathbf{b^2 = a^2 + c^2}$.
N.B. General notation – side **a** opposite to angle **A**, and so on.

Eg (i) In the triangle **ABC** above, if **AB = 3cm** and **BC = 4cm**, we can find **AC** using Pythagoras's Theorem:

$\mathbf{AC^2 = BC^2 + AB^2}$

$\mathbf{AC^2 = 4^2 + 3^2 = 16 + 9 = 25}$

$\mathbf{AC = \sqrt{25} = 5cm}$.

Eg (ii) In the triangle **ABC** above, if **AB = 3cm** and **AC = 5cm**, we can find **BC** using Pythagoras's Theorem:

$\mathbf{AC^2 = BC^2 + AB^2}$

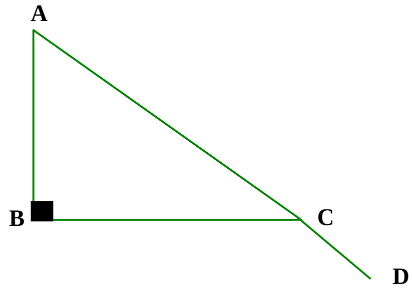
$\mathbf{5^2 = BC^2 + 3^2}$

$\mathbf{25 = BC^2 + 9}$

$\mathbf{BC^2 = 25 - 9 = 16}$

$\mathbf{BC = \sqrt{16} = 4cm}$.

Eg (iii)In the diagram below, **AB** is **perpendicular** to **BC**, **AD** is a **straight line**, **AB = 6m**, **BC = 8m** and **AD = 12m**. Find **CD**.



$AC^2 = BC^2 + AB^2$ (**AC** is the **hypotenuse**)

$AC^2 = 6^2 + 8^2$

$AC^2 = 36 + 64$

$AC^2 = 100$

$AC = \sqrt{100} = 10$m.

CD = AD - AC

CD = 12 – 10 = 2m.

Quadratic equation - solution formula

The general form of the **quadratic** function is
$y = ax^2 + bx + c$, where **a** is the coefficient of the **squared** term, **b** is the coefficient of the **middle** term and **c** is the **constant**.
The solution formula for the solution of $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

E.g.(i) Solve $2x^2 - 3x + 1 = 0$, using the quadratic solution formula.

We have **a** = **+ 2**, **b = - 3** and **c = + 1**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{+3 \pm \sqrt{+9-8}}{4}$$

$$\Rightarrow \quad x = 1 \text{ or } x = \frac{1}{2}.$$

E.g.(ii) Solve $3x^2 + 5x = 2$.

This gives $3x^2 + 5x - 2 = 0$ as the **quadratic** equation.

Then **a** = **+ 3**, **b = + 5** and **c = - 2**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-5 \pm \sqrt{+25+24}}{6}$$

$$\Rightarrow \quad x = -2 \text{ or } x = \frac{1}{3}.$$

Quadratic function

The general form of the **quadratic** function is
$y = ax^2 + bx + c$, where **a** is the coefficient of the **squared** term, **b** is the coefficient of the **middle** term and **c** is the **constant**.

Examples:

(i) **$y = 2x^2 - 3x + 1$** is a quadratic; **a** = **+ 2**, **b = - 3** and **c = + 1**.

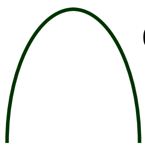
(ii) **$y = 3x^2 - 2x$** is a quadratic; **a** = **+ 2**, **b = - 2** and **c = 0**.

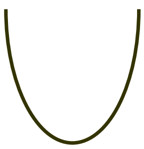
(iii) **$y = 4x^2 - 1$** is a quadratic; **a** = **+ 4**, **b = 0** and **c = - 1**.

(iv) **$y = x^2$** is a quadratic; **a** = **+ 1**, **b = 0** and **c = 0**.

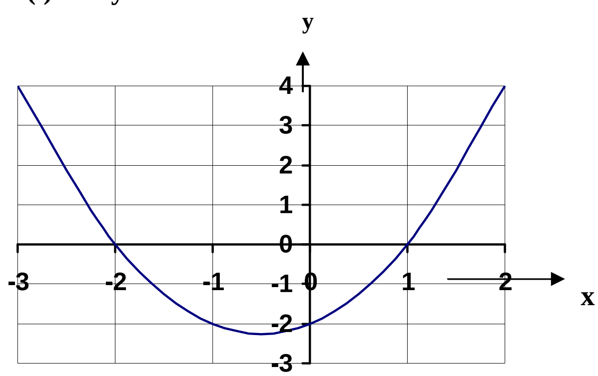
Quadratic graph

The **graph** of the **quadratic** function is a **parabola**:

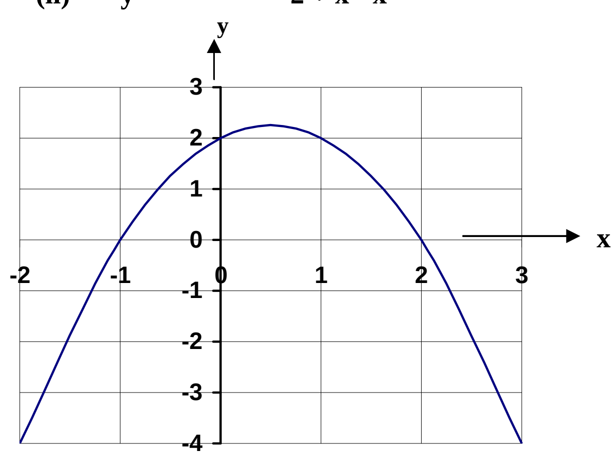

('**cap**') or


('**cup**')

(i) **y = $x^2 + x - 2$**



(ii) **y = $2 + x - x^2$**



Radian

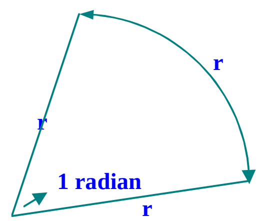
A **radian** is the **angle** at the **centre** of a **circle** that subtends an **arc equal** in length to the **radius**.

Since there are **2π radii** in the full circumference, **2π radians = 360°**

Þ **π radians = 180°**

\ **1 radian =** $\frac{180}{p}$°

and **1° =** $\frac{p}{180}$ **radians**.



Range

The **range** of a set of data is the difference between the **highest** and **lowest values** in the set.

E.g. Six children, playing a game, score:
5, 10, 4, 9, 4 and 7 points respectively.
Find the **range** of their scores.

Since the **lowest value** is **4** and the **highest value** is **10**, the **range** is **10 - 4** = **6** points.

Rational number

A rational number is any number that can be written as a **fraction**, $\frac{numerator}{denominator}$, for example 0.2, 0.2, 23%, … are rational numbers.

Reciprocal function

Any value of **x** that gives **0** as a **denominator** cannot be used as **division** by **0** is **undefined** (intuitively it is infinity).

Examples of reciprocal functions:

(i) **y** = $\frac{1}{x}$, (**x** ¹ **0**).

(ii) **y** = $\frac{1}{x^2}$, (**x** ¹ **0**).

(iii) **y** = $\frac{2}{1 - x}$, (**x** ¹ **1**).

Reciprocal function - graph

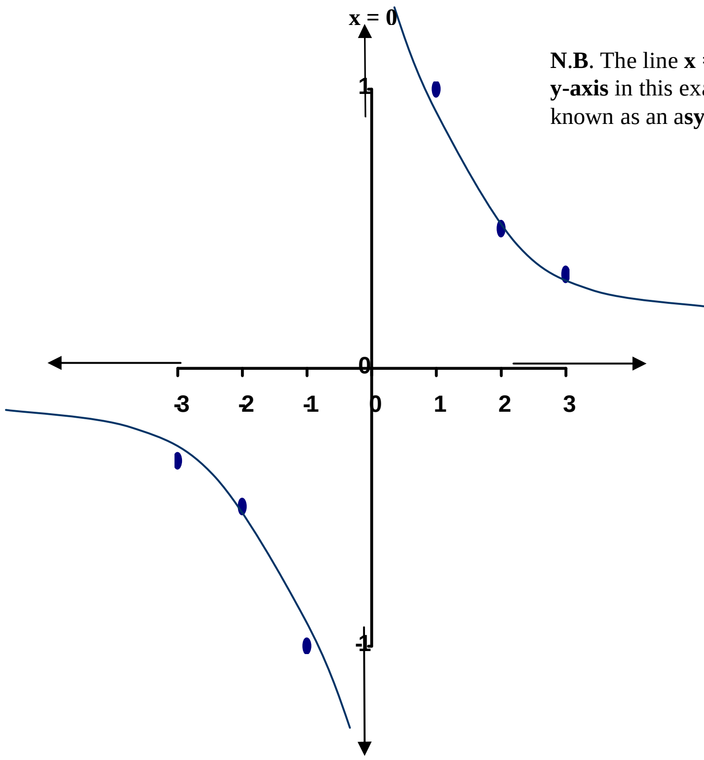
The **graph** of the reciprocal function is in **two sections**, with one section to the **left** of the **x-value** that gives a denominator of **0** and the other section to the **right** of it.

The graphs of:

	(i)	y	$=$	$\frac{1}{x}$,	$(x\ {}^{1}\ 0)$
	(ii)	y	$=$	$\frac{1}{x^2}$,	$(x\ {}^{1}\ 0)$
and	**(iii)**	y	$=$	$\frac{1}{x^2}$	$(x\ {}^{1}\ 0)$

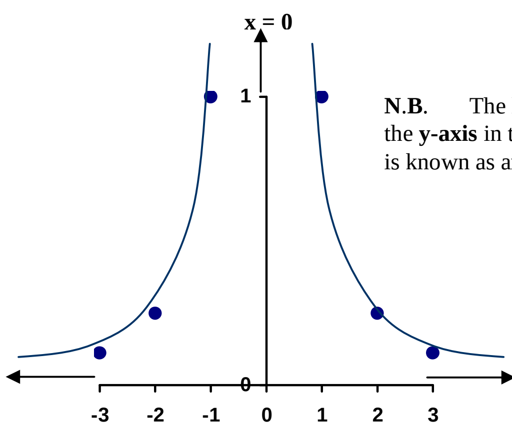
are shown on the next pages.

(i) $y = \frac{1}{x}, \quad (x \neq 0).$



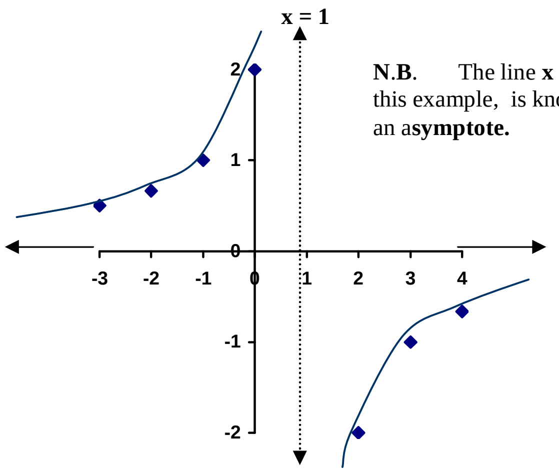
N.B. The line **x = o**, i.e. the **y-axis** in this example, is known as an a**symptote.**

(ii) $y = \frac{1}{x^2}, \quad (x \neq 0).$



N.B. The line **x = o**, i.e. the **y-axis** in this example, is known as an a**symptote.**

(iii) **y** = $\frac{2}{1 - x}$, ($x \neq 1$).



N.B. The line **x = 1** in this example, is known as an a**symptote.**

Reflection

When a **point** (or **set** of **points** forming a **shape**) is **reflected in a line** a **mirror image** is produced; each point is 'thrown' **straight across** the 'mirror line', called the **axis of reflection**, and its image lies the **same distance** away from the axis of reflection on the **other side**.

E.g. The quadrilateral **ABCD** with **vertices** **(-2, 3)**, **(2, 1)**, **(3, -2**) and **(-1, -3)** respectively has

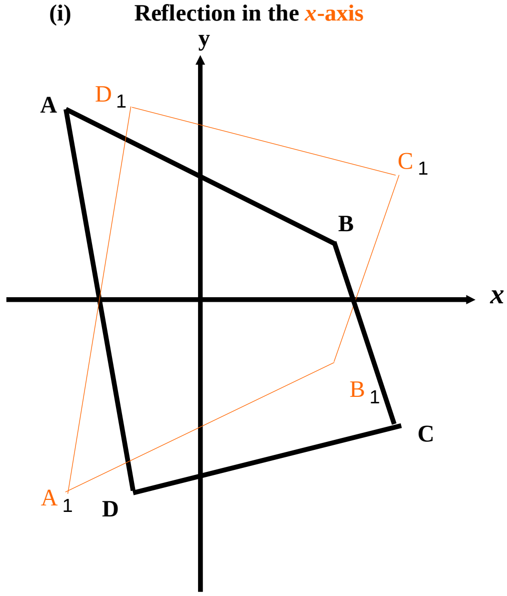
(i) $A_1 B_1 C_1 D_1$ as its **image**, following reflection in the x-axis

(ii) $A_2 B_2 C_2 D_2$ as its **image**, following reflection in the y-axis

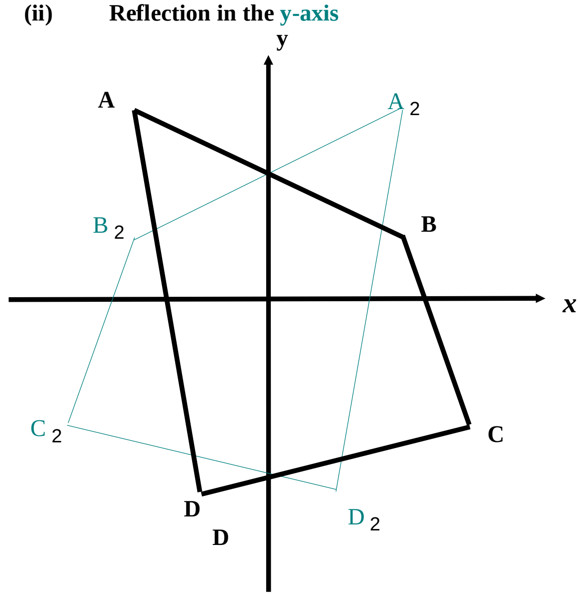
(iii) $A_3 B_3 C_3 D_3$ as its **image**, following reflection in the line y = x

(iv) $A_4 B_4 C_4 D_4$ as its **image**, following reflection in the line y = -x.

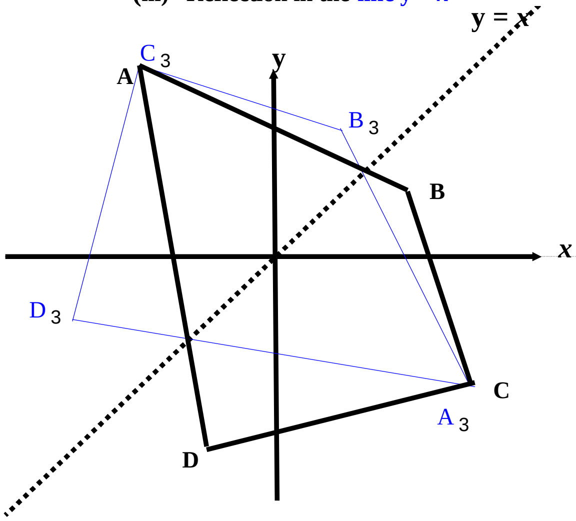
(i) **Reflection in the *x*-axis**



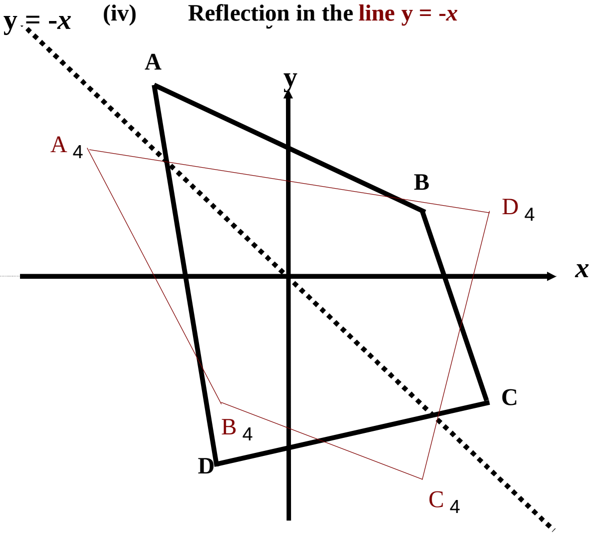
(ii) **Reflection in the y-axis**



(iii) Reflection in the line y = x



(iv) Reflection in the line y = -x



N.B. The **inverse** of a **reflection** is **reflection** of the **image** in the **same line**.

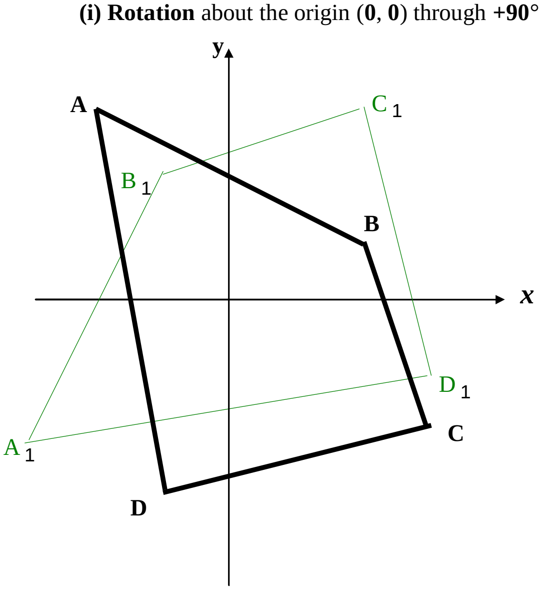
Rotation

A **rotation** is a **turning** of a **point** (or **set** of **points** forming a **shape**) through **x°** in a **clockwise** or **anti-clockwise** direction around a **centre**.

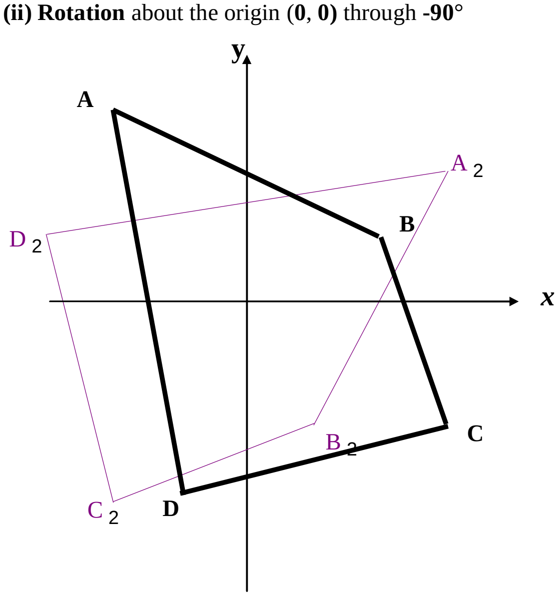
E.g. The quadrilateral **ABCD** with **vertices** **(-2, 3)**, **(2, 1)**, **(3, -2)** and **(-1, -3)** respectively has

- **(i)** $A_1 B_1 C_1 D_1$ as its **image**, following a **rotation** through **90° anti-clockwise** using the point (**0, 0**) as **centre**;
- **(ii)** $A_2 B_2 C_2 D_2$ as its **image**, following a **rotation** through **90° clockwise** using the point (**0, 0**) as **centre**;
- **(iii)** $A_3 B_3 C_3 D_3$ as its **image**, following a **rotation** through **180°** using the point (**0, 0**) as **centre**.

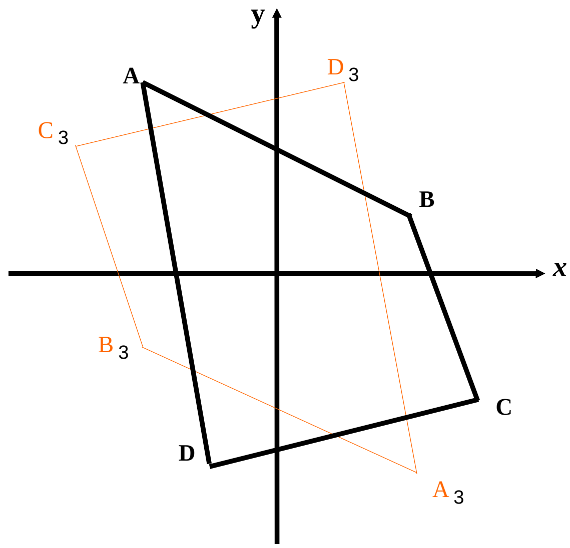
(i) Rotation about the origin (**0**, **0**) through **+90°**



(ii) Rotation about the origin (**0**, **0)** through **-90°**



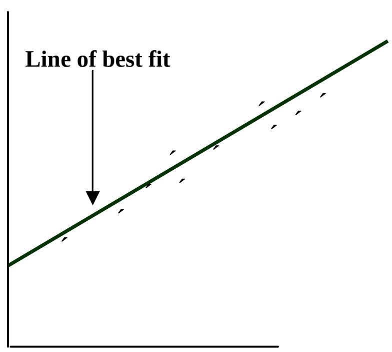
(iii) Rotation about the origin (**0**, **0**) through **180°**



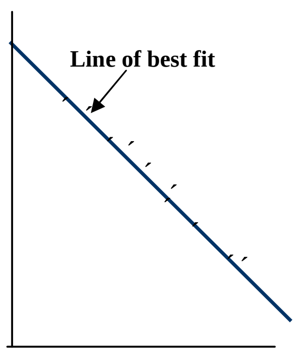
N.B. The **inverse** of **a rotation** is **rotation** in the **opposite direction** through the **same angle** and using the **same centre**.

Scatter diagram A scatter diagram is a graph used to determine whether there exists a **linear relationship** between **two variables**.

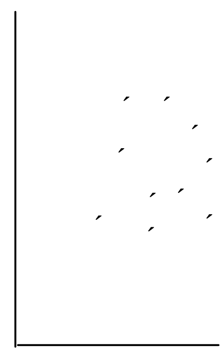
Positive Correlation
- i.e. positive gradient



Negative Correlation
- i.e. negative gradient



No Correlation

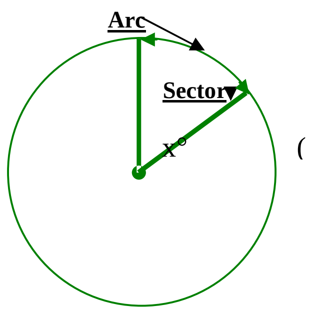


Sector of Circle

1. **The angle in the sector is measured in degrees**

(i) Length of arc $= 2\pi r \times \frac{x}{360}$

(ii) Area of sector $= \pi r^2 \times \frac{x}{360}$



(i) The sector shown has arc length:

$2 \times 3.142 \times 2 \times \frac{72}{360} =$ **2.5136 cm**.

(ii) The sector shown has area:

$3.142 \times 2^2 \times \frac{72}{360} =$ **2.5136 cm^2**.

2. **The angle in the sector (i.e. x) is measured in radians**

(i) Length of arc = rx

(ii) Area of sector $= \frac{1}{2} r^2 x$

(See entry for **Radian**.)

Sequences

A **sequence** is an arrangement, in successive order, of numbers that follow a **set pattern**. Symbolic notation may be used to express rules of sequences.

E.g. The first five terms of Sequence E are:

e_1	e_2	e_3	e_4	e_5	.	.	.	e_n
2	4	6	8	10	.	.	.	2n.

Clearly, the nth term is 2n, since E is the set of

EVEN NUMBERS **(i.e. $e_n = 2n$)**.

Worked Examples on Sequences:

(i) Find the **nth term** in the sequence: 1, 3, 5, 7, 9, …

METHOD

n	(0)	1	2	3	4	5	
	(-1)	**1**	**3**	**5**	**7**	**9**	
1st difference	**+2**	**+2**	**+2**	**+2**	**+2**		So this is **linear**.

The **nth** term is, therefore, a **straight line** whose **gradient** is **+2** and intercept point on the y-axis is (**0**,**-1**).

∴ The **nth term** is therefore: **2n-1**.

(ii) Find the **nth term** in the sequence: 2, 3, 5, 8, 12, …

METHOD

n	(0)	1	2	3	4	5	
	(2)	**2**	**3**	**5**	**8**	**12**	
1st difference	**+0**	**+1**	**+2**	**+3**	**+4**		So this is **not linear**.
2nd difference	**+1**	**+1**	**+1**	**+1**			So this is **quadratic**.

The **nth** term is, therefore, a **quadratic function**.

∴ $T_n = an^2 + bn + c$:

n	0	1	2	3	4	5
$an^2 + bn + c$	(2)	**2**	**3**	**5**	**8**	**12**

(0,2) Þ $a(0^2) + b(0) + c = 2$
⇒ $c = 2.$

(1,2) Þ $a(1^2) + b(1) + 2 = 2$
Þ $a + b = 0$ ¼ (i)

(2,3) Þ $a(2^2) + b(2) + 2 = 3$
Þ $4a + 2b = 1$ ¼ (ii)

(i) ´ 2 ¼ $2a + 2b = 0$ ¼ (iii)

(ii) – (iii) ¼ $2a = 1$

Þ $a = \frac{1}{2}$

Substitute in (i): $\frac{1}{2} + b = 0$

\ $b = -\frac{1}{2}$

The nth term is, therefore: $\frac{1}{2}n^2 - \frac{1}{2}n + 2$, as before.

Significant figure

The **first significant** figure is the **first non-zero** figure in the number. However, any **zeros included** between significant figures ***are*** themselves **significant figures**.
Also, any **zeros** that 'happen' as the result of a **rounding upwards** of **9** *are* **significant** figures.
Zeros whose role is to **preserve** the **size** of a number ***are not*** **significant figures**.

The **examples** below highlight the main points to be borne in mind:
907.059

- rounded off to **2 decimal places** is **907.06**
- rounded off to **2 significant figures** is **910**.

0.00106

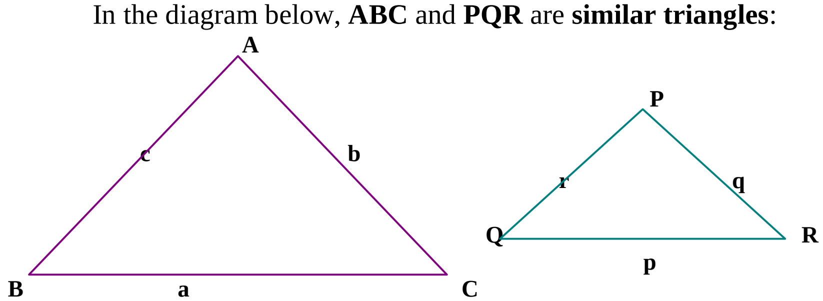
- rounded off to **2 decimal places** is **0.00**
- rounded off to **2 significant figures** is **0.0011**.

17692216.035

- rounded off to **2 decimal places** is **17692216.04**
- rounded off to **2 significant figures** is **18000000**.

Similar triangles

Triangles that are **equi-angular** are **similar**.
Similar triangles have **two** important **properties**, namely:
(i) Their **corresponding sides** are in **proportion**.
(ii) Their **areas** are to each other as the **squares** on their **corresponding sides**.
In the diagram below, **ABC** and **PQR** are **similar triangles**:



Ð **A** = Ð **P**, Ð **B** = Ð **Q**, Ð **C** = Ð **R** and

since the **corresponding sides** are **opposite** to the **equal angles**, the **three pairs** of **corresponding sides** are:
a and **p**, **b** and **q** and **c** and **r**.

Then we have:

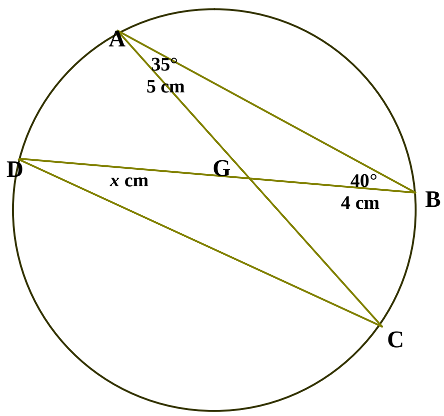
$$\frac{\mathbf{a}}{\mathbf{p}} = \frac{\mathbf{b}}{\mathbf{q}} = \frac{\mathbf{c}}{\mathbf{r}} \quad \text{and}$$

$$\frac{\mathbf{AreaABC}}{\mathbf{AreaPQR}} = \frac{\mathbf{a}^2}{\mathbf{p}^2} = \frac{\mathbf{b}^2}{\mathbf{q}^2} = \frac{\mathbf{c}^2}{\mathbf{r}^2}.$$

If we are required to **prove** that two triangles are similar, it is sufficient to find that t hey satisfy **any one** of the following **conditions**:

(i) **Two angles** in **one** triangle **equal** to **two angles** in the **other**.
(ii) **Two sides** of **one** triangle **proportional** to **two sides** in the **other**, with the **angles included** between these two sides **equal**.
(iii) **Three sides** of **one** triangle **proportional** to the **three sides** in the **other**.

E.g. Examine the diagram below and find:
(i) **x** (i.e. the **length** of **BC**).
(ii) The **ratio** of the **area** of triangle **ABG** to the **area** of triangle **DGE**.
(iii) If the **area** of triangle **DGE** is **y** cm ^{2}, find the **area** of triangle **ABG** in terms of **y**.



(i) Ð **A** = Ð **D = 35°** **(Angles in the same arc are equal.)**
Ð **B** = Ð **E = 40°** **(Angles in the same arc are equal.)**
\ Ð **AGB** = Ð **DGE = 105°** **(Third angle in each triangle.)**
Since triangles **AGB** and **DGE** are **equi-angular**, they are **similar**, so their **corresponding sides** are in **proportion**.
We must sort out the **corresponding sides** by taking the **sides opposite** to the **equal angles**.
Clearly, then:
GB corresponds to **GE** (Both are opposite to 35°.)
AG corresponds to **DG** (Both are opposite to 40°.)
AB corresponds to **DE** (Both are opposite to 105°**.**)
We have:

$$\frac{GB}{GE} = \frac{AG}{DG} = \frac{AB}{DE}.$$

To find **x** (i.e. **DG**) we use:

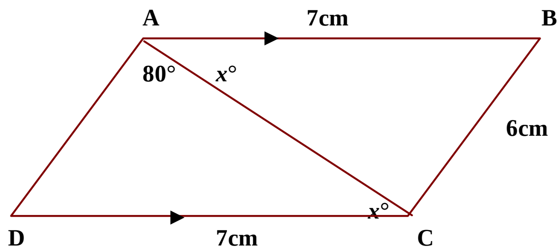
$$\frac{GB}{GE} = \frac{AG}{DG}.$$

$$\backslash \quad \frac{4}{6} = \frac{5}{x}$$

$\Rightarrow$ **4x** = **30** (By cross-multiplication)
and **x** = **7.5 cm.**

(ii) $$\frac{AreaABG}{AreaDGE} = \frac{4^2}{6^2} = \frac{16}{36} = \frac{4}{9}.$$

(iii) If **Area DGE** = **y cm** ^{2},
then **Area ABG** = $\frac{4}{9}$ **y cm** ^{2}.



In the diagram above, **AB** and **CD** are parallel and each has length **7cm**. If **BC** = **6cm** and Ð**DAC** = **80°**, prove that the triangles **ADC** and **ABC** are congruent.

Q **AB** is parallel to **DC**, Ð **BAC** = Ð **ACD** (alternate angles**)**.
Also, **Q** **AB** = **BC** (given) and **AC** is common to both triangles **ADC** and **ABC**, we have **two sides** and the **included angle** in **ADC** equal to two sides and the included angle in **ABC**.

\ **ADC** º **ABC** (**Q** Condition **SAS** is satisfied).
Q.E.D.

Simple Interest

Interest is the profit earned from an investment. If money is borrowed, then the person who borrows it must pay interest to the lender. The money that is invested or lent is called the *principal*. The percentage return per annum is the *rate %*. The length of life (in years) of the loan or investment is called the *time*. The total of the interest added to the principal is called the *amount*. The amount is, therefore, the total sum remaining invested after a certain number of years. With simple interest, the *principal stays the same*, regardless of the length of time for the investment or loan.

Eg.1 £200 is invested for 3 years at 10% per annum simple interest. Find the amount at the end of the 3 years.

	Principal	=	**£200**				
	Rate	=	**10%**				
	Time	=	**3 years**				
	Interest	=	**10%** of **£200**	´	**3**		
		=	**£20**	´	**3**	=	**£60**.
\	**Amount**	=	**£200 + £60**	=	**£260**.		

<u>**Simple Interest Formula**</u>:

P	=	**Principal**
R	=	**Rate %**
Y	=	**No. of Years**
I	=	**Interest**

$$\mathbf{I} = \frac{PRY}{100}$$

The **Simple Interest Formula** is convenient to use in calculations.
Applying the formula to the example above, we have:

P = **£200**

R = **10 %**

Y = **3**

$$I = \frac{200 \times 10 \times 3}{100} = \frac{6000}{100} = £60.$$

$\therefore$ **Amount = £200 + £60** = **£260** (as before.)

The **Simple Interest Formula** can be **transposed** to give:

P	=	$\frac{100I}{RY}$
R	=	$\frac{100I}{PY}$
Y	=	$\frac{100I}{PR}$

Eg. 2 Find the **sum of money** which needs to be invested at 10% per annum simple interest to earn £60 interest at the end of 3 years.

$$P = \frac{100 \times 60}{10 \times 3} = \frac{6000}{30} = £200.$$

Eg. 3 Find the **rate %** per annum simple interest at which £200 would have to be invested for 3 years to earn £60 interest.

$$R = \frac{100 \times 60}{200 \times 3} = \frac{6000}{600} = 10\%.$$

Eg. 4 Find the **time** it would take £200 to earn £60 interest if it is invested at 10% per annum simple interest.

$$Y = \frac{100 \times 60}{200 \times 10} = \frac{6000}{2000} = 3 \text{ years}.$$

Simpsons's Rule

The area under a curve can be approximated using **Simpson's Rule**. Using this rule, the area must be divided into an **even** number of **strips**, thereby making an **odd** number of **ordinates**.

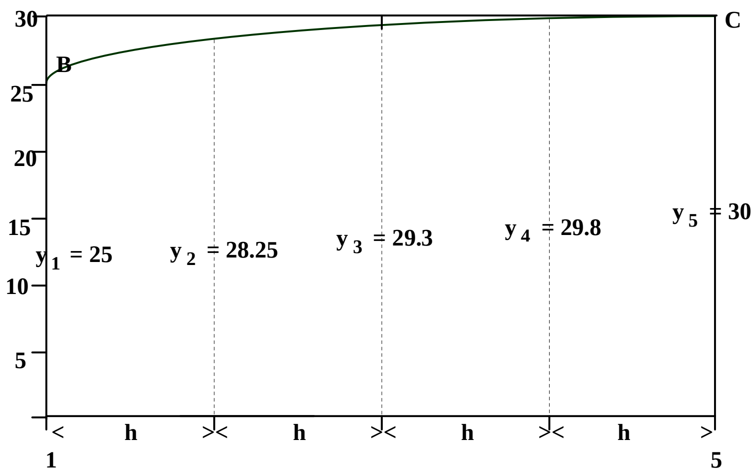
The **smaller** the value of **h** chosen, the more accurate the area obtained using this method will be.

Formula for this rule:

$$\frac{1}{3}h\,(y_1 + 4y_2 + 2y_3 + 4y_4 + \ldots 4y_{n-1} + y_n).$$

In Words:

Divide the area into an **even no.** of strips, each of interval **h**.
To the **sum** of the **first and last** ordinates, add **twice the sum of the other odd ordinates** and **four times the sum of the even ordinates**; **one third** of the **product** of **this sum** and the **interval width h** gives an approximation to the area under the curve.



Using **Simpson's Rule**, each part of the curve is treated as part of a **parabola**,
$y = ax^2 + bx + c$.

We are required to find the area under the curve **BC**.

Using **Simpson's Rule**, the area must be divided into an **even** number of **strips**, thereby making an **odd** number of **ordinates**.

The ordinates are as follows:

$y_1 = 25$

$y_2 = 28.25$

$y_3 = 29.3$

$y_4 = 29.8$

$y_5 = 30$.

The **total area** is:

$$\frac{1}{3}h\,(y_1 + 4y_2 + 2y_3 + 4y_4 + y_5)$$

$$= \quad \frac{1}{3}\left(\frac{5}{4}\right)\{25 + 4(28.25) + 2(29.3) + 4(29.8) + 30\}$$

$$= \quad \frac{5}{12}(345.8)$$

$= \quad$ **144.08** square units.

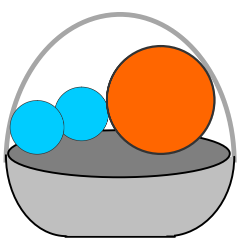
Simultaneous equations

If we have 2 equations, each containing 2 unknown quantities **x** and **y**, the solution to the system of equations is (**x**, **y**), satisfying both equations '*at the same time*'.
Equations such as these are called **simultaneous** equations.
If we have **two** equations, each containing **2 unknown quantities**, it is possible to find a **solution** that satisfies **both equations simultaneously**, which means '**at the same time'**.
Equations of this type are called **'Simultaneous Equations'**.

Eg. If **1 bat** and **2 balls** together cost **£2.00** and
2 bats and **1 ball** together cost **£2.50**, find the cost of:
(a) 1 bat and
(b) 1 ball.

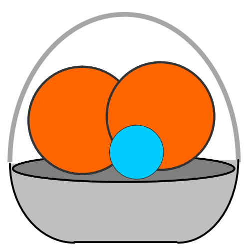
If we put the bats and balls into baskets, we have:

Cost = £2.00



Basket 1

Cost = £2.50



Basket 2

We need to have the **same number of bats** or the **same number of balls** in **each basket**. What can we do?

We **do not** know the individual prices of the bats and balls but we **do** know that we would be allowed to, say, **double** the **contents** of a basket, provided that we **also double** the **cost** of its contents.

In this case, this is what we **must** do before we can solve the problem.

We could make the **same number** of **bats** in each basket,
if we **double** the contents of **Basket 1**.
Alternatively, we could make the **same number** of **balls** in each if we **double** the contents **of Basket 2**.

Doubling Basket 2 gives: **4 bats** and **2 balls** cost **£5.00**.
We shall call this **Basket 3** .

Now, the **difference** in the **contents** between
Basket 3 and **Basket 1** is **3 bats** and
the **difference** in the **costs** is **£5 - £2** = **£3**.

Therefore, **1 bat** costs **£1.00** and **1 ball** costs **50p**.

ANSWER: **(a)** **1 bat** costs **£1.00** and **(b)** **1 ball** costs **50p**.

In the same way it is possible to solve simultaneous equations containing 3 or more unknown quantities when 3 (or more) equations are given.

Methods for **solution** of simultaneous equations include:
(i) **Elimination**
(ii) **Substitution**.

(i) **Solution of Simultaneous Equations – Elimination Method**

The method used to solve the simultaneous equations in the **example** above is really a process of **elimination**.
Notice that we were able to **eliminate** (i.e. "get rid of ")
something from the baskets by **subtraction** of the **contents**,
when we had **two baskets** containing the **same number** of it.

Whilst displaying the information in baskets helps greatly in grasping an understanding of this topic, the task is just as easy to perform (and much faster!), if we use proper **equations**, **instead** of baskets, to contain the information.
The idea is, of course, just the same. We can pretend that the equations are baskets if we want to visualise the problem!

We shall reconsider the example above, using the **algebraic** method of **elimination**, instead of the baskets, to **solve** the equations.

Letting x represent the cost, in **pence**, of a **bat** and
y represent the cost, in **pence**, of a **ball**, we have:

$$1x + 2y = 200 \quad \ldots \quad \textbf{Equation 1}$$
$$2x + 1y = 250 \quad \ldots \quad \textbf{Equation 2}$$

We are free to eliminate either x or y, but in order to do so, we must first have the **same number** of the x or y in **both equations**.
Clearly, we would have $2x$ in both equations, if we multiplied **Equation 1** by **2**. Alternatively, we could multiply **Equation 2** by **2** to give $2y$ in both equations. The amount of work is equal in either case.

Equation 1 $\times$ 2:

$$2x + 4y = 400 \quad \ldots \quad \textbf{Equation 3}$$
$$2x + 1y = 250 \quad \ldots \quad \textbf{Equation 2}$$

Equation 3 – Equation 2:

$$2x - 2x + 4y - 1y = 400 - 250$$

Simplifying gives: $3y = 150$

Divide by 3: $y = 50$.

Now substitute y = 50 in Equation 2:

$$2x + 50 = 250$$

Subtract 50: $\underline{\;-50} = \underline{-50}$

$$2x = 200$$

Divide by 2: $x = 100$.

We have now **solved** the **simultaneous equations**:

$2x + 4y = 400$

and $2x + y = 250$,

finding the **solution** to be:

$x = 100$ or **£1.00**

and $y = 50$**p**.

This means that a **bat** costs **£1.00** and a **ball** costs **50p**, as before.

Simultaneous Equations – Positive and Negative Values of x and y

The practical example of simultaneous equations considered earlier contained only **positive (**i.e. **+) values** of ***x*** and ***y***, obviously because it would **not** make sense to have **negative (**i.e. **-) values**; we would **not** have **negative numbers** of **objects**,**£**, **people**, **animals**, etc.

We shall now look at **solution** of **simultaneous equations**, involving both **positive** and **negative** values of ***x*** and ***y***.

N.B. **<u>Rules for Eliminating</u>**

Signs the **same** (i.e. **both** + or **both** -) ⇒ **subtract** the equations.
Signs **different** (i.e. **one** + and **one** -) ⇒ **add** the equations.

Example:

$$2x - y = -4 \quad \ldots \quad \text{(i)}$$
$$3x + 2y = 1 \quad \ldots \quad \text{(ii)}$$

Method:

Using **elimination** (of **y**), we have:

(i) $\times$ 2… $4x - 2y = -8$ … (iii)

(ii) + (iii)… $7x = -7$

⇒ $x = -1$.

Substituting x = -1 in equation (i):

$2(-1) - y = -4$

⇒ $-2 - y = -4$

⇒ $-2 + 4 = y$

⇒ $2 = y$.

∴ **x = -1**, **y = 2** is the solution.

Using **substitution**:
From equation (i), y = 2x + 4.
Substituting y = 2x + 4 in Equation (ii):

$$3x + 2(2x + 4) = 1$$
$$\Rightarrow \quad 3x + 4x + 8 = 1$$
$$\Rightarrow \quad 7x + 8 = 1$$
$$\Rightarrow \quad 7x = 1 - 8$$
$$\Rightarrow \quad 7x = -7$$
$$\Rightarrow \quad x = -1.$$

Substituting x = -1 in Equation (i):

$$2(-1) - y = -4$$
$$\Rightarrow \quad y = 2$$

∴ **x = -1**, **y = 2** is the solution.

N.B. The solution (**-1**, **2**) gives the **point** of **intersection** between the **2 lines**, 2x – y = - 4 and 3x + 2y = 1.

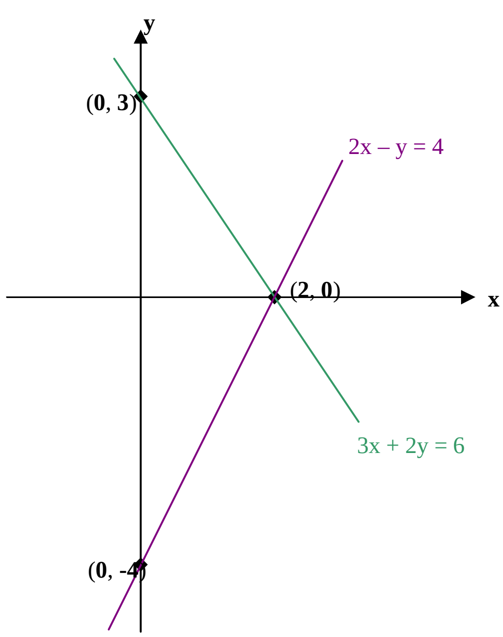
Solution of Simultaneous Equations – Graphical Method

Simultaneous Equations may be solved graphically by using intersecting graphs.
If two functions are plotted on the same axes, using the same scales, the functions are equal to each other at the point(s) of intersection between their graphs.

E.g. Solve

$$2x - y = 4 \quad \ldots \quad (i)$$
$$3x + 2y = 6 \quad \ldots \quad (ii)$$

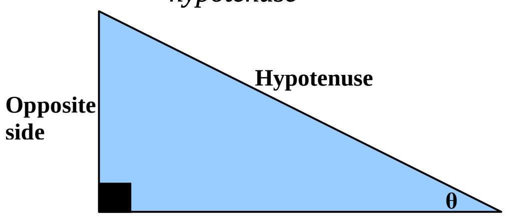
by drawing a graph.



The solution is **x = 2** and **y = 0**.

Sin(e) **Sin**(e) is the **trigonometric ratio** $\frac{opposite}{hypotenuse}$ in a **right - angled triangle**.

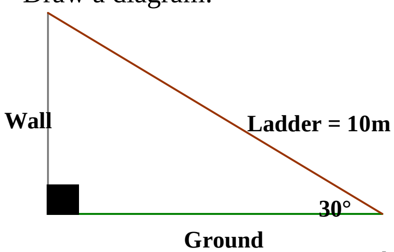
Sin θ = $\frac{opposite}{hypotenuse}$ - see diagram below:



Worked Example (using sine):

A ladder, **10m** long, is placed against a wall at an angle of **60°** to the ground. Find how far up the wall the ladder reaches

Draw a diagram:

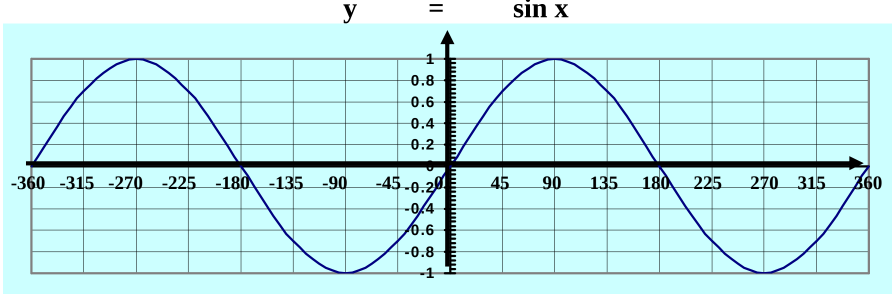


	Sin 30°	=	$\frac{wall}{ladder}$
⇒	$\frac{1}{2}$	=	$\frac{wall}{10}$
⇒	2 × wall	=	10
\	**wall**	=	**5m**.

Sine Graph:

Since **0°**, **360°**, **720°**, …all have the same **sine**, as do **180°**, **540°**, … and so do **90°**, **450°**, … . the sine graph starts to **repeat** after every **360°**; it is said to have **period 360°**.

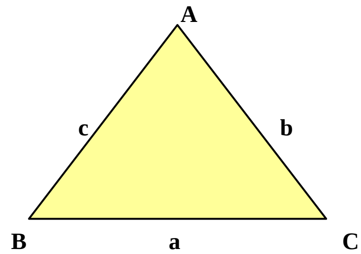
y = sin x



N.B. Notice how the graph of **y = sin x** oscillates about the ***x*-axis** between **y = 1** and **y = -1**.

Sine rule

The **sine rule (**along with the **cosine rule** - see **cosine rule** also) is used to solve **non-right-angled triangles**.



The **Sine Rule** states:

$$\frac{\mathbf{a}}{\mathbf{SinA}} = \frac{\mathbf{b}}{\mathbf{SinB}} = \frac{\mathbf{c}}{\mathbf{SinC}}.$$

NOTE: Any one of these gives the length of the diameter of the circumscribing circle of Triangle ABC.

When applying the **sine rule** to find a particular angle it is important to remember that the **sine** of an **angle** is **equal** to the **sine** of its **supplement**.

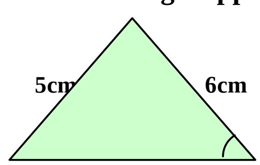
E.g. **30°** and **150°** both have the **same** sine, **0.5**.

Obviously, this can give rise to ambiguity in certain cases, so check that the correct size of angle is chosen:

E.g. If **sin A = 0.707**,
angle **A** may be **45°** or **135°** (i.e. **180° - 45°**).

The **sine rule** may be used when the following information is given:

- **Two sides** and an **angle opposite** to **one** of them:



Using the sine rule to find x°, we have:

$$\frac{5}{\sin 50°} = \frac{6}{\sin x°}$$

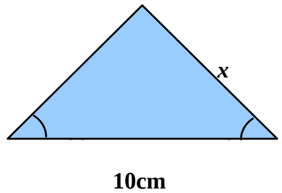
Þ **sin x° = 0.9193**

\ **x° = 66.8°**.

(x° could normally be <u>180° - 66.8</u>°, but <u>not</u> <u>here</u> since the <u>sum</u> of the <u>three</u> <u>angles</u> in a triangle is <u>180</u>°).

A **further application** of the **sine rule** would solve the triangle completely.

- **One side** and **any two angles**:



The **third** angle is **100°**.

Using the sine rule to find ***x***, we have:

$$\frac{x}{\sin 60°} = \frac{10}{\sin 100°}$$

$$\Rightarrow \quad x = 8.8\text{cm}.$$

Again, a **further application** of the **sine rule** would solve the triangle completely.

Spearman's Coefficient of Rank Correlation:

$r_s = 1 - \frac{6\sum d_i^2}{n^3 - n}$, where $d = x_i - y_i$, the **difference** in the values of the **ranks** between pairs.

Worked Example 2:

Calculate **Spearman's rank correlation coefficient** for the following data:

x	**25**	**27**	**27**	**28**	**29**	**31**	**32**	**33**	**34**	**34**
y	**45**	**49**	**51**	**54**	**52**	**60**	**60**	**62**	**63**	**64**

Method:

The sets of values for **x** and **y** must first be ranked. Where two or more *equal* values occur, the rank assigned to **each** is the **average** of the **positions occupied** by the **tied values**. Then we have:

x	rank	y	rank	point (x, y)	difference in ranks (d)	d^2
34	$1\frac{1}{2}$	64	1	(25, 45)	0	0
34	$1\frac{1}{2}$	63	2	(27, 49)	$\frac{1}{2}$	$\frac{1}{4}$
33	3	62	3	(27, 51)	$-\frac{1}{2}$	$\frac{1}{4}$
32	4	60	$4\frac{1}{2}$	(28, 54)	-1	1
31	5	60	$4\frac{1}{2}$	(29, 52)	1	1
29	6	54	6	(31, 60)	$-\frac{1}{2}$	$\frac{1}{4}$
28	7	52	7	(32, 60)	$\frac{1}{2}$	$\frac{1}{4}$
27	$8\frac{1}{2}$	51	8	(33, 62)	0	0
27	$8\frac{1}{2}$	49	9	(34, 63)	$\frac{1}{2}$	$\frac{1}{4}$
25	10	45	10	(34, 64)	$-\frac{1}{2}$	$\frac{1}{4}$

$$\sum d^2 = 3\frac{1}{2}$$

$$r_s = 1 - \frac{6\sum d_i^2}{n^3 - n} \Rightarrow r_s = 1 - \frac{6 \times 3.5}{1000 - 10} = \frac{323}{330}.$$

N.B. **-1** represents a perfect **negative** correlation and **+1** a perfect **positive** correlation – this clearly shows a strong **positive** correlation.)

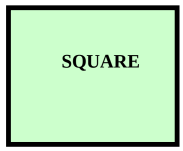
Square number

Square numbers are called this because they give the **areas** of **squares** of **edge** **1**, **2**, **3**, . . .

We have: $1^2 = 1 \times 1 = 1$

$2^2 = 2 \times 2 = 4$

$3^2 = 3 \times 3 = 9$ and so on.



Square numbers are often referred to as **squares**.

Square Numbers = {1, 4, 9, 16, 25, 36, 49, 64, . . }.

Standard deviation (GCSE Additional and Advanced Subsidiary Statistics.)

The most commonly used measure of dispersion is the **standard deviation**.

- The standard deviation is the **square root** of the **variance**.

Standard Deviation = $\sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$ or $\sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}$.

Worked Example:

Calculate the **mean** and **standard deviation** of the **temperatures** recorded over the **20-**day period in the summer, detailed earlier.
Find also what **percentage** of the temperatures is within **1 standard deviation** of the mean.

1. **Mean** = $\frac{396}{20}$ = **19.8° C**.

Score x	**Frequency** f	**Deviation from Mean** x – 19.8	$(x - 19.8)^2$	$f(x - 19.8)^2$
15	2	-4.8	23.04	46.08
18	4	-1.8	3.24	12.96
19	1	-0.8	0.64	0.64
20	6	+0.2	0.04	0.24
21	3	+1.2	1.44	4.32
22	2	+2.2	4.84	9.68
23	1	+3.2	10.24	10.24
25	1	+5.2	27.04	27.04

Sum of squared deviations: $\Sigma f(x - 19.8)^2$ = **111.2**

Sum of frequency: Σf = **20**

Standard Deviation: $\sqrt{\frac{\Sigma f(x - 19.8)^2}{\Sigma f}} = \sqrt{\frac{111.2}{20}} =$ **2.36°**.

Temperatures within the range **19.8°** ± **2.36°**,
i.e. **17.44°** to **22.16°**, we have **16** out of **20** i.e. **80%** of the scores.

Standard form (or Standard Index Notation)

Large and small numbers can both be written conveniently as:

A $\quad = \quad$ **10^{n}**, where $\mathbf{1 \le A < 10}$ and **n** is an **integer**.

This is known as **Standard Form** (or **Standard Index Notation**).

Worked examples:

Write the following numbers in standard form, correct to
3 significant figures:

(i) 6752
(ii) 0.0006752
(iii) 2186000
(iv) 0.2186
(v) 6.387

Answers:

(i) $6752 = 6.752 \times 10^{3} = \mathbf{6.75 \times 10^{3}}$ (to **3 sig. figs.**)

(ii) $0.0006752 = 6.752 \times 10^{-4} = \mathbf{6.75 \times 10^{-4}}$ (to **3 sig. figs.**)

(iii) $2186000 = 2.186 \times 10^{6} = \mathbf{2.19 \times 10^{6}}$ (to **3 sig. figs.**)

(iv) $0.2186 = 2.186 \times 10^{-1} = \mathbf{2.19 \times 10^{-1}}$ (to **3 sig. figs.**)

(v) $6.387 = 6.387 \times 10^{0} = \mathbf{6.39 \times 10^{0}}$ (to **3 sig. figs.**)

Stem and leaf diagram

Worked Example:

A class of twelve pupils were given tests in French and Spanish and the percentage marks are recorded in the table below.
Use a stem and leaf diagram to compare their performance in these subjects.

French	12 39 42 46 58 62 69 71 72 73 82 98	**Mean 60.3**
Spanish	39 41 46 51 53 54 63 67 69 78 79 81	**Mean 60.1**

French			**Test Marks (%)**		**Spanish**	
		8	9			
		2	8	1		
3	2	1	7	8	9	
	9	2	6	3	7	9
		8	5	1	3	4
	6	2	4	1	6	
		9	3	9		
		2	1			

N.B. The scores are ordered and recorded, in descending order, back to back from the central 'stem', reading backwards on the left and forwards on the right of the stem. Look at the third row where all the marks are seventy-something; the 71, 72 and 73 for French branch out from the central stem backwards and the 78 and 79 for Spanish branch out from the central stem forwards. (Separate diagrams, each with stem and right 'half-leaf', are often used.)

E.g. On the second-last row 9/3 reads 39 in French and 3/9 reads 39 in Spanish also.

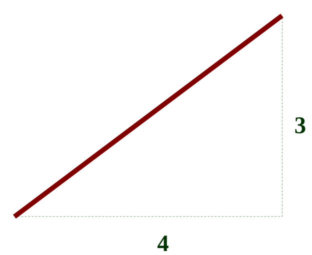
Straight line function

The general form of the **straight line** function is
y = mx + c, where **m** is the **gradient** (or *slope*) and **c** is the **intercept** on the **y-axis**.

The **gradient** of a line is equivalent to the **tan**(gent) in trigonometry. It can be conveniently thought of as the $\frac{\textbf{Rise}}{\textbf{Run}}$, as is used in builders' language.

A line **sloping from the left upwards** has a **positive gradient**.

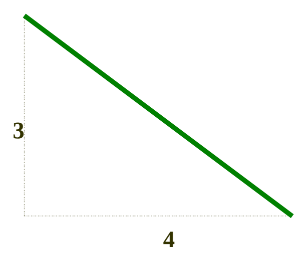
For example, the line below has gradient $\frac{+3}{4}$.



RUN 4 units to the **right** and **RISE 3** units.

A line **sloping from the left downwards** has a **negative** gradient.

For example, the line below has gradient $\frac{-3}{4}$.



RUN 4 units to the **left** and **RISE 3** units.

Examples:

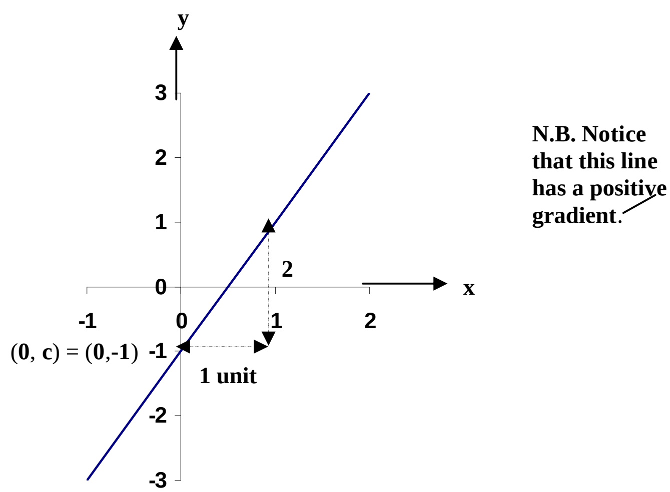
(i) **y = 2x – 1** is a straight line; **m** = + $\frac{2}{1}$ and **c = – 1**.

(ii) **2x + 3y – 6 = 0** is also a straight line.

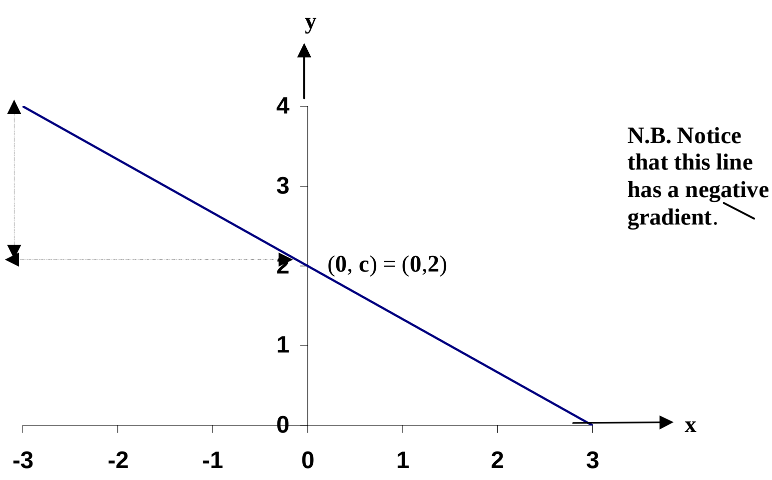
Writing it in the form **y = mx + c** gives **m = -** $\frac{2}{3}$, **c = + 2**.

Straight line graph

(i) **Sketch** of **y = 2x – 1**



(ii) **Sketch** of **y** = $-\frac{2}{3}x + 2$



Surd Quantities such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, … are **irrational numbers**. If they are left like this, without being worked out, they are known as **surds**. It is usually convenient, and indeed more accurate, to manipulate surds without working them out.

Tally chart

E.g. In a survey, **20** children were asked how many magazines they read in each week.
The results of the survey were as follows:

2	6	0	4	1	3	4	2	0	4
2	4	1	1	7	3	4	5	6	2

We organise raw data by listing the different scores in **ascending order of size** and then drawing up a **TALLY CHART**, as follows:

Score	Tally	Frequency
0	II	2
1	III	3
2	IIII	4
3	II	2
4	~~IIII~~	5
5	I	1
6	II	2
7	I	1

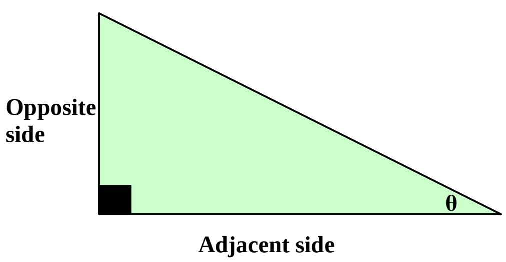
N.B. IIII = 4
but ~~IIII~~ **= 5**

Tan(gent)

(See **Differentiation** and **Straight Line**.)

Tan(gent) is the **trigonometric ratio** $\frac{opposite}{adjacent}$ in a **right - angled triangle**.

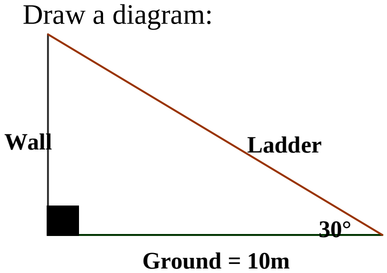
Tan θ = $\frac{opposite}{adjacent}$ **-** see diagram below:



Worked Example (using tan):

A ladder is placed against a wall at an angle of **30°** to the ground.
If the foot of the ladder rests at a distance of **10m** from the wall, find how far up the wall the ladder reaches

Draw a diagram:

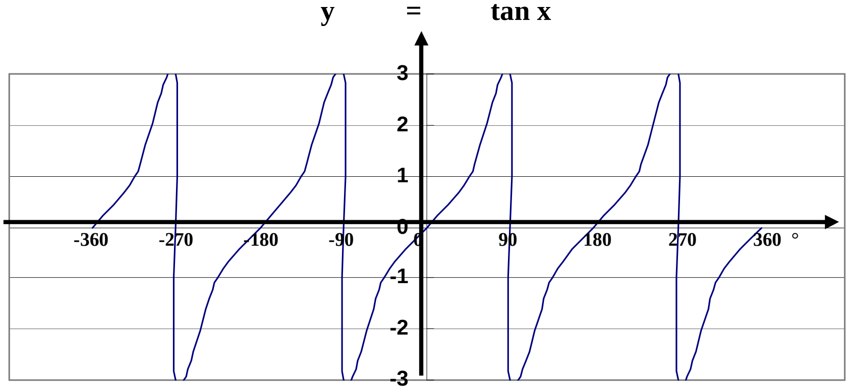


	Tan 30°	=	$\frac{wall}{ground}$
⇒	0.5773502	=	$\frac{wall}{10}$
⇒	wall	=	$10 \times 0.577\ldots$
\	**wall**	**=**	**5.8m**. (Correct to 1 decimal place.)

Tan Graph:

Since **0°**, **360°**, **720°**, …all have the same **tan**, as do **180°**, **540°**, … and so do **90°**, **450°**, … the tan graph starts to **repeat** after every **180°**; it is said to have **period 180°**.

y = tan x



N.B. The graph of **y** is piecewise. Since **y** is **undefined** for **x** as any **odd multiple** of **90°**, it **tends towards infinity** at those points.

Gradient of Tangent:

The **gradient** of the **tangent** to a curve at *any* point on the curve can be found using **differentiation**:

$\frac{dy}{dx}$ gives a **general expression** for the **gradient** of the **tangent** to a curve at *any* point on the curve. All that is required to find the **actual gradient** at a **particular point** is to substitute the value of **x** at *that point* into $\frac{dy}{dx}$.

E.g. If $y = 2x^2 + x - 1$ find the **equation** of the **tangent** to the curve at the point **(1, 2)**.

$y = 2x^2 + x - 1$

Þ $\frac{dy}{dx} = 4x + 1$

x = 1 Þ $\frac{dy}{dx} = 4(1) + 1 = 5$.

\ the **gradient** of the curve at the point where **x = 1** is **5**.

Q the **gradient** of the **tangent** is also **5**, we have:

y = mx + c *(Remember the tangent is a straight line.)*

(1, 2) and **m = 5** ⇒ **2 = 5(1) + c**

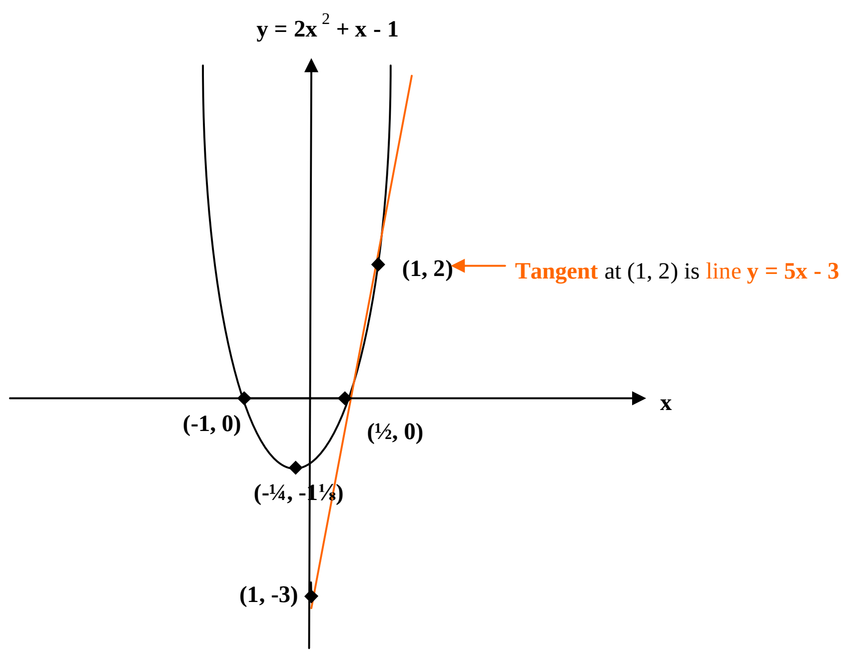
⇒ **2 = 5 + c**

⇒ **-3 = c**

\ **y = 5x – 3** is the equation of the **tangent** to the curve,

$y = 2x^2 + x - 1$, at the point **(1, 2)**.

See the diagram below.



Terminating decimal number

A **terminating decimal** number is a rational number,

e.g. $1.56 = 1\frac{56}{100} = \frac{156}{100}$.

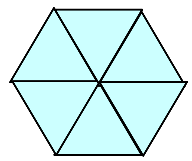
Tessellation

(Latin: ***tessera*** 'a small tile used in mosaics')

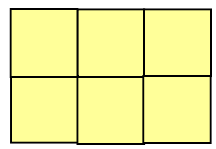
A tessellation is a tiling.

The **regular polygons** that will tessellate are:

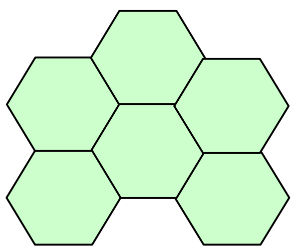
Equilateral Triangle



Square



Regular Hexagon



Translation (Latin: '*trans'* means 'across' and '*latum'* means 'brought'.)

Translation through $\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$.

A translation is simply a **displacement** of a **point** (or **set** of **points** forming a **shape**) through $\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$.

The ***x*** **component** of the translation is **added** to the ***x*-coordinate** of the point and the **y component** of the translation is **added** to the **y coordinate** of the point to give the **coordinates** of the **new position** of the point.

A **translation** $\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$ can be likened to $\begin{pmatrix}\textbf{Easting}\\ \textbf{Northing}\end{pmatrix}$.

E.g. A point transformed by a **translation** $\begin{pmatrix} \mathbf{2} \\ \mathbf{3} \end{pmatrix}$ moves **2 units** to the **East** and **3 units** to the **North**.

A **minus Easting** then, is a **Westing** and a **minus Northing** is a **Southing**, so a point transformed by a **translation** $\begin{pmatrix} \mathbf{-2} \\ \mathbf{-3} \end{pmatrix}$ is moved **2 units** to the **West** and **3 units South**.

It follows that a **translation** $\begin{pmatrix} \mathbf{-2} \\ \mathbf{3} \end{pmatrix}$ moves a point **2 units** to the **West** and **3 units North** and a **translation** $\begin{pmatrix} \mathbf{2} \\ \mathbf{-3} \end{pmatrix}$ moves a point **2 units** to the **East** and **3 units** to the **South**.

It is convenient to write the **coordinates** as a column vector
i.e. in the **same form** as the **translation**, $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$,
and simply **add** their **corresponding elements** to find the **image**
(i.e. the **coordinates** of the **new position** of the point).

E.g. If we wish to transform a point **P (-1, 3)** by a **translation**,
T = $\begin{pmatrix} \mathbf{-2} \\ \mathbf{4} \end{pmatrix}$ we have:

$$\begin{matrix} \mathbf{P} & & \mathbf{T} & & \mathbf{P_1} \\ \begin{pmatrix} \mathbf{-1} \\ \mathbf{3} \end{pmatrix} & + & \begin{pmatrix} \mathbf{-2} \\ \mathbf{4} \end{pmatrix} & = & \begin{pmatrix} \mathbf{-3} \\ \mathbf{7} \end{pmatrix} \end{matrix}$$

where $\mathbf{P_1}$ **(-3, 7)** is the **new position** of **P** under the **translation T**.

The quadrilateral **ABCD** with **vertices (-2, 3)**, **(2, 1)**, **(3, -2**) and **(-1, -3)** respectively has $A_1B_1C_1D_1$ as its **image**, following a **translation T =** $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$. (See the figure below.)

Taking **each vertex** of **ABCD**, one at a time, we have:

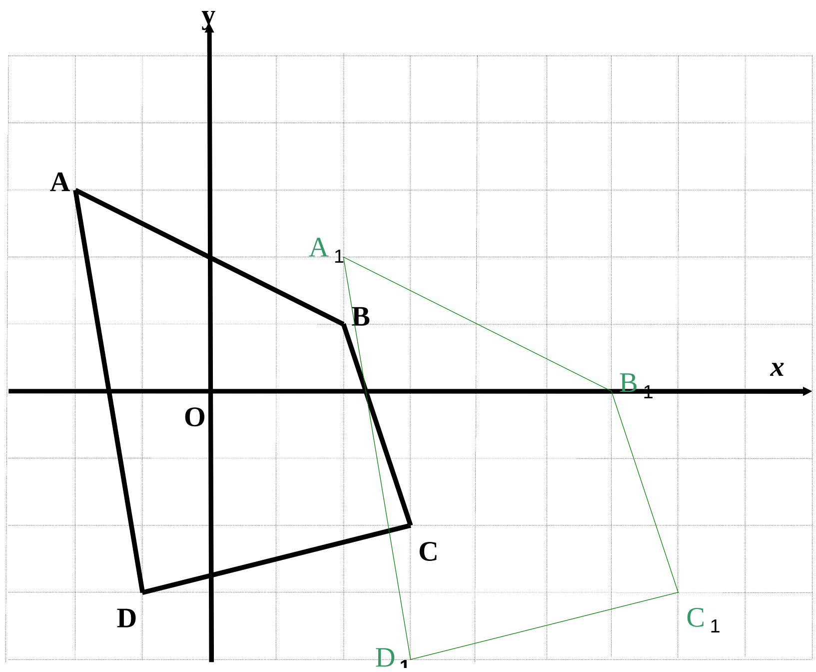
$$\underset{A}{\begin{pmatrix} -2 \\ 3 \end{pmatrix}} + \underset{T}{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} = \underset{A_1}{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}$$

$$\underset{B}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} + \underset{T}{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} = \underset{B_1}{\begin{pmatrix} 6 \\ 0 \end{pmatrix}}$$

$$\underset{C}{\begin{pmatrix} 3 \\ -2 \end{pmatrix}} + \underset{T}{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} = \underset{C_1}{\begin{pmatrix} 7 \\ -3 \end{pmatrix}}$$

$$\underset{D}{\begin{pmatrix} -1 \\ -3 \end{pmatrix}} + \underset{T}{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} = \underset{D_1}{\begin{pmatrix} 3 \\ -4 \end{pmatrix}}$$

Notice that the **translation** $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ would move $A_1B_1C_1D_1$ **back** to **ABCD** i.e. **changing** the **signs** of **both elements** of the **translation** gives the **INVERSE** translation.

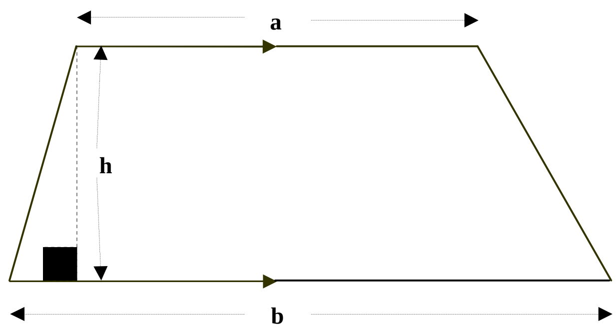


Figure

Trapezium A trapezium is a **quadrilateral** with **one pair** of its sides **parallel** to each other.

Area of a **trapezium**: $\frac{1}{2}$ **Sum of Parallel Sides** ´ **Perpendicular Height**.

Formula for **area** of **trapezium**: $\frac{1}{2}h(a + b)$ (Se figure below.)



Trapezium (or Trapezoidal Rule

The area under a curve can be approximated using the Trapezium Rule. When the area is divided up into a number of strips, **each** of width **h**, each strip is near enough to being a **trapezium**.
Obviously, the **smaller** the value of **h** chosen, the closer the strip will be to a trapezium, and, therefore, the area obtained using this method can be more and more accurate.
The **parallel sides** of the 'trapezia' are the **ordinates**,

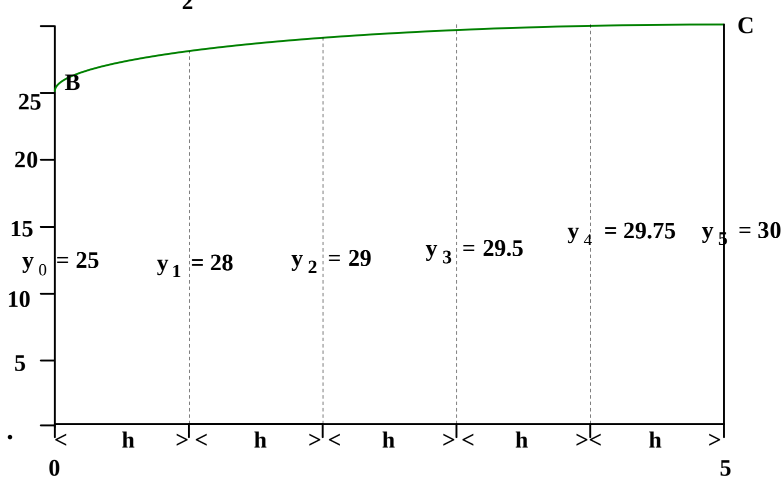
y_0, y_1, y_2, y_3, y_4, y_5, etc.

In words, the Trapezium Rule is:

[$\frac{1}{2}$ (**1**st **+ last**) + **sum** of *everything* **in between**] ´ **h**.

When all this is simplified, we obtain a **formula** for this rule:

$\frac{1}{2}h(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$.



We are required to find the area under the curve **BC**.

Divide the area up into a **convenient number** of strips, each of width **h**, and notice that each strip is near enough to being a **trapezium**, i.e. a quadrilateral with **one pair** of **parallel** sides. Obviously, the **smaller** the value of **h** chosen, the closer the strip will be to a trapezium, and, therefore, the area obtained using this method can be more and more accurate.

Remember the **formula** for the **area** of a **trapezium**:

$\frac{1}{2}$ **Sum of Parallel Sides ´ Perpendicular Height**.

The **parallel sides** of our ‘trapezia’ are the **ordinates**,

y_0, y_1, y_2, y_3, y_4 and y_5.

Then, using The Trapezoidal Rule, we have:

STRIP NO.	AREA	UNIT ^{2}
1	$\frac{1}{2}(y_0 + y_1)$´ $h = \frac{1}{2}(25 + 28)$ ´ 1	26.5
2	$\frac{1}{2}(y_1 + y_2)$´ $h = \frac{1}{2}(28 + 29)$ ´ 1	28.5
3	$\frac{1}{2}(y_2 + y_3)$´ $h = \frac{1}{2}(29+29.5)$ ´ 1	29.25
4	$\frac{1}{2}(y_3 + y_4)$´ $h = \frac{1}{2}(29.5+29.75)$ ´ 1	29.63
5	$\frac{1}{2}(y_4 + y_5)$´ $h = \frac{1}{2}(29.75+30)$ ´ 1	29.88
	\ the total area under the curve is approximately	<u>143.76</u>.

The <u>total area</u>:

$\frac{1}{2}(y_0 + y_1)$ ´ h + $\frac{1}{2}(y_1 + y_2)$ ´ h + $\frac{1}{2}(y_2 + y_3)$ ´ h + $\frac{1}{2}(y_3 + y_4)$ ´ h + $\frac{1}{2}(y_4 + y_5)$ ´ h

can be <u>factorised</u> to give:

$\frac{1}{2} h (y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5)$.

<u>The general formula for The Trapezoidal Rule</u>:

$\frac{1}{2} h (y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$.

Trial & improvement method for solving equations

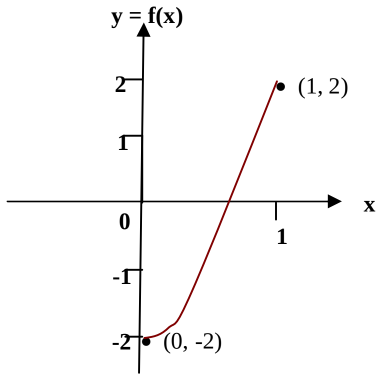
E.g. By trial and improvement, find a positive value for **x** satisfying the equation $3x^2 + x - 2 = 0$, correct to 1 decimal place.

When **x = 1**, $f(x) = 3(1^2) + 1 - 2 = 2$ ¹ **0**.

When **x = 0**, $f(x) = 3(0^2) + 0 - 2 = -2$ ¹ **0**.

\ **f(x)** crosses the **x-axis** somewhere between **x = 0** and **x = 1**.

The ‘picture’ looks something like this:



When **x = 0.5**, **f(x)** = $3(0.5^2) + 0.5 - 2$ = **- 0.75**. (Too small).

When **x = 0.6**, **f(x)** = $3(0.6^2) + 0.6 - 2$ = **- 0.32**. (Too small).

When **x = 0.7**, **f(x)** = $3(0.7^2) + 0.7 - 2$ = **+ 0.17**. (Too big).

Q **f(0.7) = + 0.17**, **f(x) = 0** somewhere between **x = 0.6** and **x = 0.7**.

Try **x = 0.65**, then **f(x)** = $3(0.65^2) + 0.65 - 2$ = **-0.0825**. (Too small).

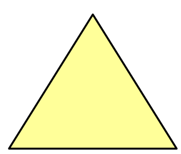
So **x = 0.7** is the solution, correct to 1 decimal place.

Triangle

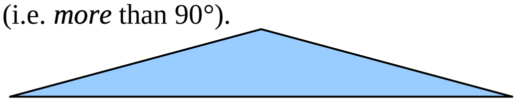
A triangle is a **closed 3 – sided** shape.
It contains **three** angles, the **sum** of which is ***always*** **180°**.

There are **different types** of triangle:

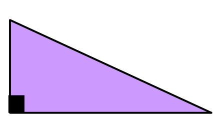
(i) **Acute** – **angled**, in which ***each*** angle is ***less*** than **90°**.



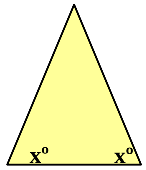
(ii) **Obtuse** – **angled**, in which ***one*** angle is **obtuse** (i.e. *more* than 90°).



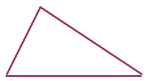
(iii) **Right** – **angled**, in which ***one*** angle is **90°**.



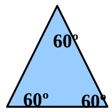
(iv) **Isosceles**, in which ***two angles*** *and* ***two sides*** are **equal**.



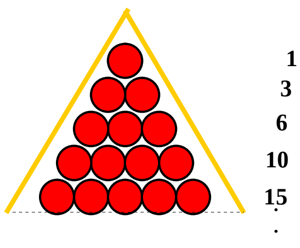
(v) **Scalene**, in which ***the three angles*** *and* ***the three sides*** are **different**.



(vi) **Equilateral**, in which ***all three angles*** are equal, making ***each angle*** equal to **60°** (i.e. 180° ÷ 3).



Triangular number



Notice how this sequence progresses **-** the **difference** between **successive terms** is **increased** by **1** each time.
We have: **add 2**, **add 3**, **add 4**, **add 5**, **add 6**, **add 7** and so on.

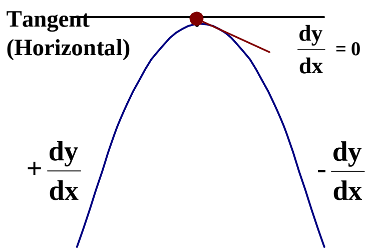
This would remind you of the 15 red balls enclosed in a triangular frame before commencement of a snooker game.

Triangular Numbers = {**1**, **3**, **6, 10**, **15**, **21**, . . . }

Turning point (GCSE Additional Pure and Advanced Subsidiary Pure)

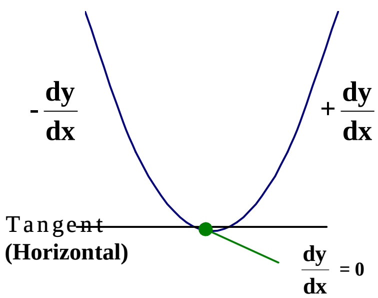
Since $\frac{dy}{dx}$ is synonymous with 'gradient', a positive $\frac{dy}{dx}$ indicates a positive **gradient and a** negative $\frac{dy}{dx}$ **indicates a** negative **gradient**.

When $\frac{dy}{dx}$ changes from **positive** to **negative**, or from **negative** to **positive,** there is an **instant** in between where the **gradient** is **neither** negative nor positive**,** i.e. $\frac{dy}{dx}$ **=** **0**.



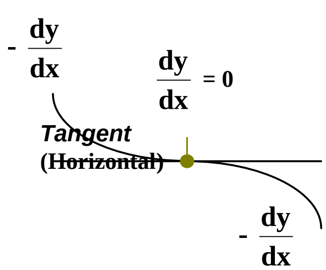
MAXIMUM TURNING POINT

Diagram 1



MINIMUM TURNING POINT

Diagram 2



POINTS OF INFLEXION

Diagram 3(a)

OR

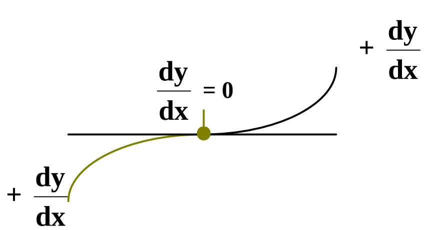


Diagram 3(b)

To find **x values** at **turning points** on a curve:

(i) Find $\frac{dy}{dx}$.

(ii) Set $\frac{dy}{dx} = 0$.

(iii) Solve for **x**.

- To determine whether a turning point is a **maximum**, a **minimum** or a **point** of **inflexion**:

  (i) Find $\frac{dy}{dx}$.

  (ii) Set $\frac{dy}{dx} = 0$.

  (iii) Solve for **x**.

  (iv) Find $\frac{d^2y}{dx^2}$.

  (v) Substitute value of **x** found at (iii) into $\frac{d^2y}{dx^2}$.

  (vi) $\frac{d^2y}{dx^2} > 0$, i.e. **positive**, implies a **minimum** turning point.

  $\frac{d^2y}{dx^2} < 0$, i.e. **negative**, implies a **maximum** turning point.

  $\frac{d^2y}{dx^2} = 0$, implies a **point** of **inflexion**

E.g. Find the **turning point** on the curve, $y = x^2 - 4$ and determine whether it is a maximum or a minimum.

(iv) $\frac{dy}{dx} = 2x$.

(v) $2x = 0$.

(vi) $2x = 0$ Þ $x = 0$.

(vii) $\frac{d^2y}{dx^2} = +2$, i.e. **positive**.

(viii) Turning point is **minimum**.

\ **(0, -4)** is **minimum turning point** on the curve, $x^2 - 4$.

Upper Bound

A number rounded off to so many decimal places or significant figures is **not** an **accurate** representation of the number - there is a **margin** of **error**.

E.g. A population of **52 million**, correct to **2 significant figures**, lies somewhere between **51.5 million** and **52.5 million**. In this population, **52.5** is the **upper bound**.

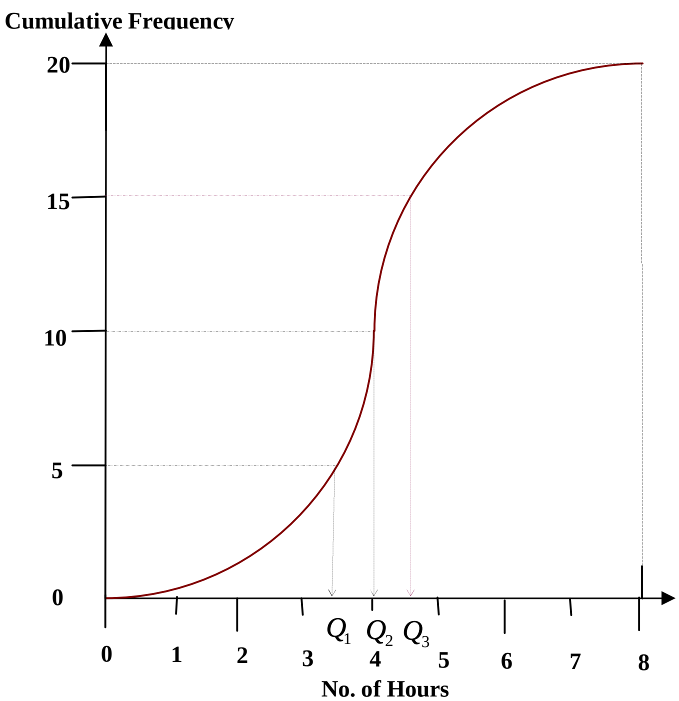
Another population is **147 million**, correct to **3 significant figures**, lies somewhere between **146.5** and **147.5 million**. In this population, **147.5** is the **upper bound**.

The **upper bound** for the **difference** between these two populations is **96 million**, i.e. **147.5 – 51.5**.

Upper quartile

When the **total frequency** on a **cumulative frequency curve** (ogive) is divided into **quarters**, the **score** corresponding to the **upper quarter** is the **upper quartile**. (See Cumulative Frequency diagram.)

On the cumulative frequency curve below, the **upper quartile**, $Q_3 = 4.6$.



Variance (GCSE Additional and Advanced Subsidiary Statistics.)

- **Formula for Variance** = $\frac{\sum (x_i - \bar{x})^2}{n} = \frac{\sum x_i^2}{n} - \bar{x}^2$.

NOTE: Please see **Standard Deviation** for more detail.

Vector

- A **vector** is a quantity that has **magnitude** (or modulus) and **direction**.
- Any point **P (*x*, y)** has **position vector** $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$, relative to the **origin O (0,0)**.
- The **magnitude** (i.e. modulus or size) of **OP** is $\sqrt{\mathbf{x}^2 + \mathbf{y}^2}$, (by **Pythagoras' Theorem**).
- A **displacement vector** or **translation** is equivalent to $\begin{pmatrix} \textbf{Easting} \\ \textbf{Northing} \end{pmatrix}$.
- Vectors may be **added** (or **subtracted**) by **adding** (or **subtracting**) their **components**:

  E.g. $\mathbf{a} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix}$

  $\Rightarrow \quad \mathbf{a} + \mathbf{b} = \begin{pmatrix} \mathbf{p} + \mathbf{r} \\ \mathbf{q} + \mathbf{s} \end{pmatrix}$

  and $\quad \mathbf{a} - \mathbf{b} = \begin{pmatrix} \mathbf{p} - \mathbf{r} \\ \mathbf{q} - \mathbf{s} \end{pmatrix}$.

- A vector may be **multiplied** by a **scalar**:

  E.g. $\mathbf{a} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} \mathbf{w} \\ \mathbf{z} \end{pmatrix}$

  $\mathbf{2a} = \begin{pmatrix} \mathbf{2x} \\ \mathbf{2y} \end{pmatrix}$

  and $\quad -\frac{\mathbf{3}}{\mathbf{2}}\mathbf{b} = \begin{pmatrix} -\frac{\mathbf{3}}{\mathbf{2}}\mathbf{w} \\ -\frac{\mathbf{3}}{\mathbf{2}}\mathbf{z} \end{pmatrix}$.

- **Base Vectors –** **i** and **j**

i and **j** are *unit vectors* (i.e. vectors of **length 1**)
in the **positive directions** of the **x** and **y** axes respectively.
It follows that a point **P** with coordinates **(x , y)** has position vector x**i** + y**j** using this notation.

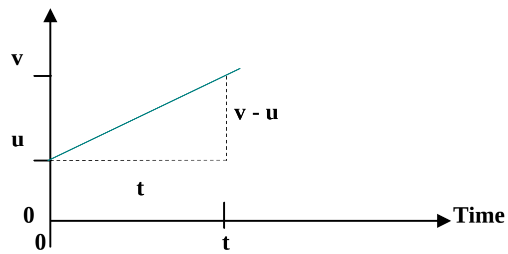
- If **p** represents the position vector **OP**, we have
  This is known as '**component'** or '**Cartesian'** form.

Velocity

Velocity is the **rate of change** of **displacement** with **time**, generally in metres per second, **m/s** or **ms** $^{-1}$. It follows that the **area** under a **velocity/time graph** gives **displacement**. This area can be found by using **integration** or one of the **area-approximating rules**, **Trapezium Rule**, **Mid-ordinate Rule** or **Simpson's Rule**.

- The **area** under a **velocity/time** graph gives the **displacement**.

E.g. **Velocity** (**Constant Acceleration**)



Let **u** be the **initial velocity** and **v** be the **velocity at time t**.

Since **area** under a **velocity/time graph** gives **displacement**, we have:

$$s = \frac{1}{2}(u + v)t \qquad \textbf{(Area of Trapezium)}.$$

When the acceleration is **variable**, the **velocity**/**time** graph is **curved**.
In this case, the **displacement** may be **approximated** by using one of the **Area-approximating Rules**
(Mid-ordinate, Trapezium or Simpson's).

The accurate way to find the **acceleration** is to **differentiate velocity** with respect to **time** i.e. **a** (acceleration) $= \frac{dv}{dt}$;
- not always an option as it can be used only when an equation of displacement in terms of time is known.

- The **gradient** of a **velocity**/**time** graph gives the **acceleration**.
When the acceleration is **variable**, the **velocity**/**time** graph is **curved**.
In this case, the gradient may be **approximated** by drawing a **tangent** to the curve at the required **time point** and calculating its gradient.

The accurate way to find the **acceleration** is to **differentiate velocity** with respect to **time** i.e. **a** (acceleration) $= \frac{dv}{dt}$;
- not always an option as it can be used only when an equation of displacement in terms of time is known.

Velocity/Time Graphs

Velocity is **speed** in a given **direction**, e.g. 80km/h North-East.

If a **velocity/time graph** is drawn:

- the **area** under the graph gives the **displacement**.
- the **gradient** of the graph gives the **acceleration/deceleration** since **acceleration** is the **rate** of **change** of **velocity** with **time**.

A **velocity/time graph** is interpreted in the following way:

- A **horizontal line** indicates a **constant speed**, i.e. there is **no acceleration** or **deceleration**.

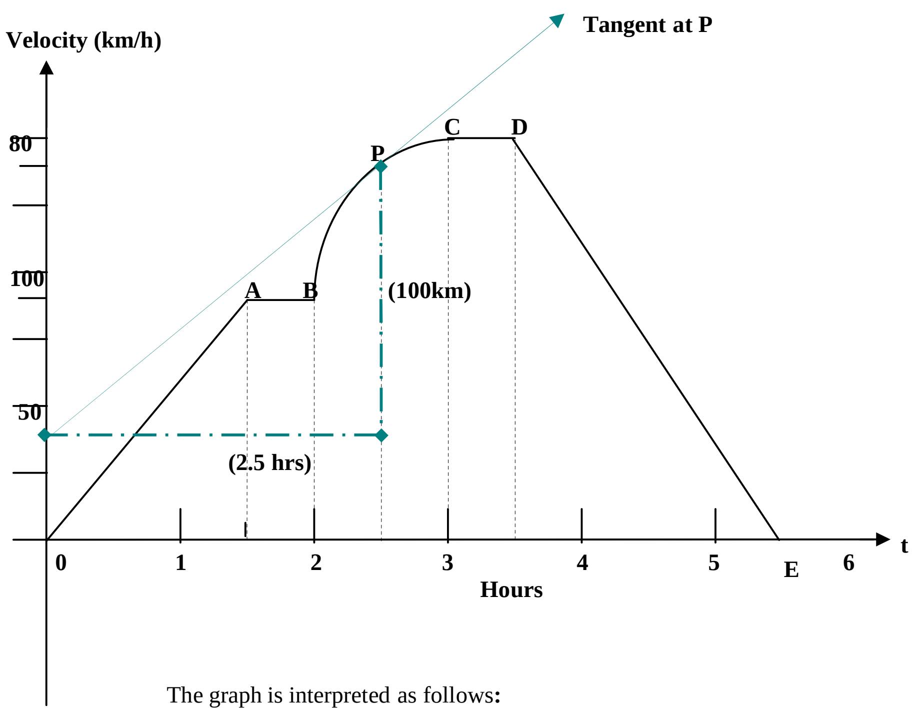
- A **straight line** sloping **upwards** towards the **right** indicates **constant acceleration**, i.e. the **gradient** is **positive**.

- A **straight line** sloping **downwards** towards the **right** indicates **constant deceleration**, i.e. the **gradient** is **negative**.

- A **curved line** indicates **non-uniform acceleration**.
  However, the **acceleration** (or **deceleration**) at any time **t** on the **curve** can be found by drawing a **tangent** to the **curve** at the **point** for **t**, and finding its **gradient**.This **gradient** gives the **acceleration/deceleration**.

- The **area** between a **velocity/time graph** and the horizontal **t-axis** gives the **displacement**. Any area bounded by **straight** lines can be found easily by using the normal formulae for **areas** of **rectangles** and **triangles** (i.e. **lb** and $\frac{1}{2}$ **bh**).
  Where there is a **curved outline** there is not always a set formula for finding the area. On these occasions, the **area** can be **approximated** by using one of the **area-approximating rules**: **the Trapezoidal Rule**, **the Mid-ordinate Rule**, or **Simpson's Rule**. When the equation of the curve is known the area under the curve can be found more accurately using integration.

The following **graph** represents the journey of a motorist from home and back.



The graph is interpreted as follows**:**

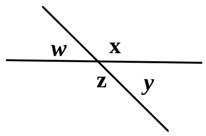
OA: A **straight** line sloping **upwards**,indicates **constant acceleration**.
AB: A **horizontal** line indicates **constant speed**.
BC: A **curved** line indicates **variable speed**.
CD: A **horizontal** line indicates **constant speed**.
DE: A **straight** line sloping **downwards**,indicates **constant deceleration**.

Vertically opposite angles

When **two lines intersect** the **opposite angles** are **equal** to each other; the opposite angles are called **vertically opposite** angles.

Angles ***w*** and ***y*** are **vertically opposite** and, therefore, **equal**.

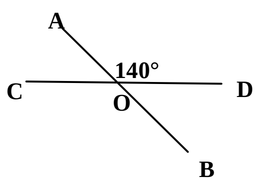
Angles ***x*** and ***z*** are **vertically opposite** and, therefore, **equal**.



Example:

AB and CD are straight lines intersecting at a point O.

Given that angle AOD is 140°, find the angles COB, AOC and BOD.

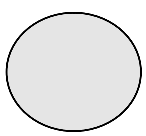


COB = AOD = 140° (Vertically opposite angles).
AOC = (180 – 140)° = 40° (Angles in a straight line).
BOD = AOC = 40° (Vertically opposite angles).

Volume

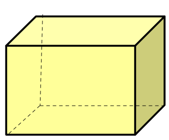
Volume is **cubic measurement** (i.e. 3-dimensional), e.g. cm^3, m^3, etc.

It is found as a result of **multiplying 3 dimensions** together.

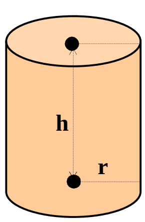


Volume of this sphere, $\frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times (\frac{1}{2})^3 = \frac{11}{21}$ cm^3.

($\mathbf{r} = \frac{1}{2}$ cm is radius. **N.B.** π is **not** a dimension; $\pi \approx 3\frac{1}{7}$.)



Volume of this cuboid, **lbh** = 7.5 cm^3.
(**l** = 2.5cm is length, **b** = 2cm is breadth and **h** = 1.5cm is height).



Volume of this cylinder, $\pi r^2 h = 7\frac{6}{7}$ cm^3.

(**r** = 1cm is radius, **h** = 2.5cm is height and $\pi = 3\frac{1}{7}$).