S6 REVISION FOR PURE MATHEMATICS

Allow me to discuss this dozen of questions with you.

1. The nth term of an arithmetic progression (A.P) is $\frac{3n-1}{6}$. Show that the sum of the first n terms of the progression is $\frac{n}{12}(3n+1)$.

   Review

   - nth term of an AP, $u_n = l = a + (n-1)d$
   - sum of n terms of an AP, $s_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + l)$
   - nth term of a GP, $u_n = ar^{n-1}$
   - sum of n terms of a GP, $s_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n - 1)}{r-1}$
   - sum to infinity, $s_\infty = \frac{a}{1-r}$

   For $u_n = \frac{3n-1}{6}$; First term, $u_1 = a = \frac{2}{6}$; second term, $u_2 = \frac{5}{6}$

   So, common difference, $d = u_2 - u_1 = \frac{5}{6} - \frac{2}{6} = \frac{1}{2}$

   Here, $s_n = \frac{n}{2}\left[2 \times \frac{1}{3} + (n-1) \times \frac{1}{2}\right] = \frac{n}{12}(3n+1)$

2. Given that $\log_2 3 = p$ and $\log_4 5 = q$, prove that $\log_{45} 2 = \frac{1}{2(p+q)}$.

   Review of laws of logarithms;

   - $\log_a b + \log_a c = \log_a bc$; $\log_a b - \log_a c = \log_a \frac{b}{c}$
   - $n \log_a b = \log_a b^n$; $\log_a b = \frac{\log_c b}{\log_c a} = \frac{1}{\log_b a}$
   - $a^{\log_a b} = b$

   For this question,

$$RHS = \frac{1}{2(p+q)} = \frac{1}{2[\log_2 3 + \log_4 5]} = \frac{1}{2\left[\log_2 3 + \dfrac{\log_2 5}{\log_2 4}\right]} = \frac{1}{2\log_2 3 + \log_2 5}$$

$$= \frac{1}{\log_2 9 + \log_2 5} = \frac{1}{\log_2 45} = \log_{45} 2 = LHS$$

3. If $e^x = \tan 2y$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{e^x - e^{3x}}{2(1+e^{2x})^2}$.

Review: chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx}$; $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{dt}{dx}$

Quotient rule; $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

For $e^x = \tan 2y$, differentiating both sides

$$e^x = 2\sec^2 2y\frac{dy}{dx} = 2(1+\tan^2 2y)\frac{dy}{dx} = 2(1+e^{2x})\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{e^x}{2(1+e^{2x})};\ \frac{d^2y}{dx^2} = \frac{(1+e^{2x})e^x - e^x\cdot 2e^{2x}}{2(1+e^{2x})^2} = \frac{e^x - e^{3x}}{2(1+e^{2x})^2}.$$

4. The curve $y = ax^2 + b\mathbf{x} + c$ has a maximum point at $(2\,,18)$ and passes through the point $(0\,,10)$. Find the values of *a*, *b* and *c*.

For turning/stationary points, $\dfrac{dy}{dx} = 0$.

$y = ax^2 + b\mathbf{x} + c$

At $(0\,,10)$, $10 = a(0) + b(0) + c$: $c = 10$

At $(2\,,18)$, $18 = 4a + 2b + 10$; $2a + b = 4 \ \cdots$(i)

Now $\dfrac{dy}{dx} = 2ax + b$

At $(2, 18)$, $\frac{dy}{dx}$ $4a + b = 0 \cdots$(ii)

Now (ii) – (i); $a = -2$; $b = 8$

5. Use the substitution $t = \tan x$ to show that $\int_0^{\frac{\pi}{4}} \frac{1}{1 + \sin 2x} dx = \frac{1}{2}$.

Recall, if $t = \tan x$, then $\sin 2x = \frac{2t}{1+t^2}$; $\cos 2x = \frac{1-t^2}{1+t^2}$; $\tan 2x = \frac{2t}{1-t^2}$.

For $t = \tan x$, $\frac{dt}{dx} = \sec^2 x = (1 + t^2) \Rightarrow dx = \frac{dt}{1+t^2}$

Also change the limits i.e.

x	0	$\frac{\pi}{4}$
t	0	1

Now $\int_0^{\frac{\pi}{4}} \frac{dx}{1 + \sin 2x} \equiv \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2}} \cdot \frac{dt}{1+t^2} = \int_0^1 \frac{dt}{(1+t)^2}$

$$= \left[\frac{-1}{1+t}\right]_0^1 = -\frac{1}{2} - -1 = \frac{1}{2}$$

6. Find the coefficient of x^{17} in the expansion of $\left(x^3 + \frac{1}{x^4}\right)^{15}$.

7. Given that $y = \ln(1 + \sin x)$, deduce that $\frac{d^2y}{dx^2} + e^{-y} = 0$.

8. Given that $\mathbf{a}=\begin{pmatrix}3\\0\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}-1\\-3\end{pmatrix}$, find the values of t such that $t\,\mathbf{a}+\mathbf{b}$ and $t\,\mathbf{b}+\mathbf{c}$ are perpendicular .

9. Show that the vectors $\begin{pmatrix}3\\4\\1\end{pmatrix}$, $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ are coplanar.

10. Find $\int \frac{dx}{x+\sqrt{x}}$.

11. Solve the equation $\sqrt{\frac{x-1}{2x}}-3\sqrt{\frac{2x}{x-1}}=2$.

12. The points $A(4,0), B(0,3)$ and $P(x\,,y)$ are such that PA and PB are always perpendicular. Show that the locus of P is a circle. Hence find the centre and radius of the circle.