S6 REVISION FOR PURE MATHEMATICS

Allow me to discuss this dozen of questions with you.

1. The nth term of an arithmetic progression (A.P) is $\frac{3n-1}{6}$. Show that the sum of the first n terms of the progression is $\frac{n}{12}(3n+1)$.

Review

- nth term of an AP, $u_n = l = a + (n-1)d$
- sum of n terms of an AP, $s_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + l)$
- nth term of a GP, $u_n = ar^{n-1}$
- sum of n terms of a GP, $s_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n - 1)}{r-1}$
- sum to infinity, $s_\infty = \frac{a}{1-r}$

For $u_n = \frac{3n-1}{6}$; First term, $u_1 = a = \frac{2}{6}$; second term, $u_2 = \frac{5}{6}$

So, common difference, $d = u_2 - u_1 = \frac{5}{6} - \frac{2}{6} = \frac{1}{2}$

Here, $s_n = \frac{n}{2}\left[2 \times \frac{1}{3} + (n-1) \times \frac{1}{2}\right] = \frac{n}{12}(3n+1)$

2. Given that $\log_2 3 = p$ and $\log_4 5 = q$, prove that $\log_{45} 2 = \frac{1}{2(p+q)}$.

Review of laws of logarithms;

- $\log_a b + \log_a c = \log_a bc$; $\log_a b - \log_a c = \log_a \frac{b}{c}$
- $n \log_a b = \log_a b^n$; $\log_a b = \frac{\log_c b}{\log_c a} = \frac{1}{\log_b a}$
- $a^{\log_a b} = b$

For this question,

$$RHS = \frac{1}{2(p+q)} = \frac{1}{2[\log_2 3 + \log_4 5]} = \frac{1}{2\left[\log_2 3 + \frac{\log_2 5}{\log_2 4}\right]} = \frac{1}{2\log_2 3 + \log_2 5}$$

$$= \frac{1}{\log_2 9 + \log_2 5} = \frac{1}{\log_2 45} = \log_{45} 2 = LHS$$

3. If $e^x = \tan 2y$, prove that $\frac{d^2y}{dx^2} = \frac{e^x - e^{3x}}{2(1+e^{2x})^2}$.

Review: chain rule, $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$; $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$

Quotient rule; $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

For $e^x = \tan 2y$, differentiating both sides

$$e^x = 2\sec^2 2y\frac{dy}{dx} = 2(1+\tan^2 2y)\frac{dy}{dx} = 2(1+e^{2x})\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{e^x}{2(1+e^{2x})};\ \frac{d^2y}{dx^2} = \frac{(1+e^{2x})e^x - e^x\cdot 2e^{2x}}{2(1+e^{2x})^2} = \frac{e^x - e^{3x}}{2(1+e^{2x})^2}.$$

4. The curve $y = ax^2 + b\mathbf{x} + c$ has a maximum point at $(2, 18)$ and passes through the point $(0, 10)$. Find the values of *a*, *b* and *c*.

For turning/stationary points, $\frac{dy}{dx} = 0$.

$y = ax^2 + b\mathbf{x} + c$

At $(0, 10)$, $10 = a(0) + b(0) + c$: $c = 10$

At $(2, 18)$, $18 = 4a + 2b + 10$; $2a + b = 4 \cdots$(i)

Now $\frac{dy}{dx} = 2ax + b$

At $(2\,,18)$, $\frac{dy}{dx}\;4a+b=0\;\cdots$(ii)

Now (ii) – (i); $a=-2$; $b=8$

5. Use the substitution $t=\tan x$ to show that $\int_0^{\frac{\pi}{4}}\frac{1}{1+\sin 2x}dx=\frac{1}{2}$.

Recall, if $t=\tan x$, then $\sin 2x=\frac{2t}{1+t^2}$; $\cos 2x=\frac{1-t^2}{1+t^2}$; $\tan 2x=\frac{2t}{1-t^2}$.

For $t=\tan x$, $\frac{dt}{dx}=\sec^2 x=\left(1+t^2\right)\Rightarrow dx=\frac{dt}{1+t^2}$

Also change the limits i.e.

X	0	$\frac{\pi}{4}$
T	0	1

Now $\int_0^{\frac{\pi}{4}}\frac{dx}{1+\sin 2x}\equiv\int_0^1\frac{1}{1+\frac{2t}{1+t^2}}\cdot\frac{dt}{1+t^2}=\int_0^1\frac{dt}{(1+t)^2}$

$$=\left[\frac{-1}{1+t}\right]_0^1=-\frac{1}{2}--1=\frac{1}{2}$$

6. Find the coefficient of x^{17} in the expansion of $\left(x^3+\frac{1}{x^4}\right)^{15}$.

Review: Expansion, $(a+b)^n=a^n+{}^nC_1.a^{n-1}b+{}^nC_2.a^{n-2}b^2+...+b^n$

The expansion for a general term;
$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$

For the $(r+1)^{th}$ term, $U_{r+1} = {}^{n}C_r\, a^{n-r} b^r$; $r = 0,1,2,...$

Now for $\left(x^3 + \frac{1}{x^4}\right)^{15}$; $n = 15$, $a = x^3$, $b = \frac{1}{x^4}$

$$U_{r+1} = {}^{15}C_r . \left(x^3\right)^{15-r} . \left(x^{-4}\right)^r = {}^{15}C_r . x^{45-7r}$$

For the term in x^{17}; $17 = 45 - 7r \Rightarrow r = 4$

Therefore the coefficient required is ${}^{15}C_4 = \frac{15!}{(15-4)!\,4!} = 1365$

7. Given that $y = \ln(1 + \sin x)$, deduce that $\frac{d^2y}{dx^2} + e^{-y} = 0$.

First apply the laws of logarithms and then the chain rule.

$$y = \ln(1 + \sin x) \Rightarrow e^y = 1 + \sin x$$

$$e^y \frac{dy}{dx} = \cos x \quad ; \quad e^y \frac{d^2y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 = -\sin x$$

Eliminating $\sin x$ and $\frac{dy}{dx}$, $e^y \frac{d^2y}{dx^2} + e^y \left(\frac{\cos x}{e^y}\right)^2 = -\left(e^y - 1\right)$

$$e^y \frac{d^2y}{dx^2} + e^y \left(\frac{1 - \sin^2 x}{e^{2y}}\right) = -\left(e^y - 1\right)$$

$$e^y \frac{d^2y}{dx^2} + e^y \left(\frac{1 - \left(e^y - 1\right)^2}{e^{2y}}\right) + \left(e^y - 1\right) = 0$$

$$e^{2y} \frac{d^2y}{dx^2} - e^{2y} + 2e^y + e^{2y} - e^y = 0$$

$$e^{2y} \frac{d^2y}{dx^2} + e^y = 0 \Rightarrow \frac{d^2y}{dx^2} + e^{-y} = 0$$

8. Given that $\mathbf{a}=\begin{pmatrix}3\\0\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}-1\\-3\end{pmatrix}$, find the values of t such that $t\,\mathbf{a}+\mathbf{b}$ and $t\,\mathbf{b}+\mathbf{c}$ are perpendicular .

$t\,\mathbf{a}+\mathbf{b}=t\begin{pmatrix}3\\0\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3t+1\\1\end{pmatrix}$; $t\,\mathbf{b}+\mathbf{c}=t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}-1\\-3\end{pmatrix}=\begin{pmatrix}t-1\\t-3\end{pmatrix}$

For perpendicular vectors $t\,\mathbf{a}+\mathbf{b}$ and $t\,\mathbf{b}+\mathbf{c}$, $(t\,\mathbf{a}+\mathbf{b})\cdot(t\,\mathbf{b}+\mathbf{c})=0$

So, $\begin{pmatrix}3t+1\\1\end{pmatrix}\cdot\begin{pmatrix}t-1\\t-3\end{pmatrix}=0 \Rightarrow (3t+1)(t-1)+1(t-3)=0$

$3t^2-t-4=0$; $(3t-4)(t+1)=0$

$$t=\frac{4}{3}\ ,\quad t=-1$$

9. Show that the vectors $\begin{pmatrix}3\\4\\1\end{pmatrix}$, $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ are coplanar.

Review, if vectors $\mathbf{a}$, $\mathbf{b}$ $and\,\mathbf{c}$ are coplanar, then $\mathbf{a}\,.(\mathbf{b}\times\mathbf{c})=0$ or $\mathbf{b}\,.(\mathbf{a}\times\mathbf{c})=0$ or $\mathbf{c}\,.(\mathbf{a}\times\mathbf{b})=0$.

$$\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}9\\-3\\-15\end{pmatrix}$$

Now $\begin{pmatrix}1\\-2\\1\end{pmatrix}.\left[\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}3\\-1\\2\end{pmatrix}\right]=\begin{pmatrix}1\\-2\\1\end{pmatrix}.\begin{pmatrix}9\\-3\\-15\end{pmatrix}=9+6+-15=0$

Since $\begin{pmatrix}1\\-2\\1\end{pmatrix}.\left[\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}3\\-1\\2\end{pmatrix}\right]=0$, the vectors are coplanar.

10. Find $\int\frac{dx}{x+\sqrt{x}}$.

Let $u = \sqrt{x}$; $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow dx = 2u\,du$

$$\int \frac{dx}{x+\sqrt{x}} \equiv \int \frac{2u\,du}{u^2+u} = \int \frac{2}{u+1}du = 2\ln(u+1) + C = 2\ln\left(1+\sqrt{x}\right) + C$$

11. Solve the equation $\sqrt{\frac{x-1}{2x}} - 3\sqrt{\frac{2x}{x-1}} = 2$.

For $\sqrt{\frac{x-1}{2x}} - 3\sqrt{\frac{2x}{x-1}} = 2$, let $a = \frac{x-1}{2x}$: "*Here notice a reciprocal*"

$\Rightarrow \sqrt{a} - \frac{3}{\sqrt{a}} = 2$; $\left(\sqrt{a}\right)^2 - 2\sqrt{a} - 3 = 0$.

$\left(\sqrt{a}-3\right)\left(\sqrt{a}+1\right) = 0$; $a = 9$ *or* $a = 1$

When $a = 9$, $\frac{x-1}{2x} = 9$; $x = -\frac{1}{17}$

When $a = 1$, $\frac{x-1}{2x} = 1$; $x = -1$

Remember to test for such an equation with surds.

For $x = -\frac{1}{17}$, $LHS = \sqrt{9} - \frac{3}{\sqrt{9}} = 3 - 1 = 1 = RHS$

For $x = -1$, $LHS = \sqrt{1} - 3\sqrt{1} = -2 \neq RHS$

$\therefore \quad x = -\frac{1}{17}$

12. The points $A(4,0), B(0,3)$ and $P(x, y)$ are such that PA and PB are always perpendicular. Show that the locus of P is a circle. Hence find the centre and radius of the circle.

Grad PA = $\frac{y-0}{x-4} = \frac{y}{x-4}$; Grad PB = $\frac{y-3}{x-0} = \frac{y-3}{x}$

Review: If two lines with gradients m_1 and m_2 are perpendicular, then $m_1 \times m_2 = -1$

For this case, $\left(\frac{y}{x-4}\right).\left(\frac{y-3}{x}\right)=-1$

$y(y-3)=-x(x-4) \Rightarrow x^2+y^2-4x-3y=0$ of the form
$x^2+y^2+2gx+2fy+c=0$, hence a circle.

By comparison; $2g=-4,\ g=-2;\ 2f=-3,\ f=-\frac{3}{2}$ and $c=0$

Centre is $(-g,-f)=\left(2,\frac{3}{2}\right)$

Radius, $r=\sqrt{g^2+f^2-c}=\sqrt{(-2)^2+\left(-\frac{3}{2}\right)^2-0}=2.5\ units$

NOTE:

(i) *You are required to continue doing practice wherever you are. The pandemic will soon come to pass.*
(ii) *Do practice all the content you were given right from s5.*
(iii) *Such series of discussions will continue, God willing.*
(iv) *I wish you God's blessings in everything you do.*
(v) *Stay home, maintain social distance and revise. Thank you.*